Surface Waves and Pattern Formation in Vibrated Granular Beds

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ABSTRACT: We present numerical simulations of surface waves in vibrated granular systems. For the simulations we use an event driven (ED) procedure with a non-constant restitution coefficient. The important advantage of a non-constant restitution coefficient is that it allows to circumvent the "inelastic collapse", i.e. the divergence of collision frequency, usually observed in ED simulations of dense and dissipative granular systems.

1 INTRODUCTION

Under vibration, granular materials show interesting phenomena like fluidization with an unexpected scaling behavior (Luding et al., 1994c; Luding et al., 1994d; Luding, 1995; Warr et al., 1995; Müller et al., 1997), or surface waves (Melo et al., 1994; Melo et al., 1995; Umbanhowar et al., 1996; Clément et al., 1996; Luding et al., 1996). For low densities the event driven (ED) simulation method is very efficient and was used to study fluidization (Luding et al., 1994c; Luding et al., 1994d; Luding, 1995). With increasing density and dissipation the ED method gets more and more ineffective due to a divergence of collision frequency, i.e. the inelastic collapse (McNamara and Young, 1994; McNamara and Young, 1996). In contrast, one observes in "soft" sphere simulations, where the contact duration $t_c > 0$ is a function of the particle stiffness, an inefficient dissipation, i.e. the so called "detachment effect" (Luding et al., 1994a).

The inelastic collapse is connected to the fact that the contact duration t_c is implicitly zero in ED. One way to handle the inelastic collapse is to allow single particles to transform into clusters, and then, to treat the clusters with the largest relative velocity (LRV) method (Luding et al., 1994c). An alternative approach that hinders the inelastic collapse was recently proposed to allow the simulation of surface waves in rather dense 2D systems (Luding et al., 1996). The idea is to introduce a cut-off time t_c, so that contacts are elastic if one of the involved particles had already a contact within time t_c. In the following we will apply this idea to simple situations and will discuss its validity with respect to simulations of granular materials. Finally, we present ED simulations in a range of parameter space inaccessible with the traditional ED algorithm.

2 SIMULATION ASPECTS

2.1 Boundary conditions

The system is made of a horizontal bottom, two vertical walls with a distance L apart, and it is open to the top. In the simulations N particles with diameter d_i (i=1, ..., N) are filled into the container. We denote the horizontal direction by x, and the gravitational acceleration g is directed into positive vertical z direction.

2.2 Event driven algorithm

In the event driven (ED) method the particles follow an undisturbed Newtonian motion under the influence of gravity, until an event occurs. An event can be either the collision of a particle with a wall or the collsion of two particles. The time to the next contact of two particles may be calculated analytically and the duration of an event is implicitly zero. We introduce dissipation on contact via the restitution coefficient ϵ, where $\epsilon = 1$ in the elastic and $\epsilon = 0$ in the completely inelastic limit. For details concerning the ED algorithm see (Luding et al., 1994d; Luding, 1995).

2.3 Collision model

For a collision we consider two particles i and j with diameters d_i, d_j, and masses m_i, m_j. The normal unit vector for their contact is $\vec{n} = (\vec{r}_i - \vec{r}_j)/|\vec{r}_i - \vec{r}_j|$, where $\vec{r}_i$ is the vector to the center of particle i. The relative velocity of the particle surfaces at the contact point is

$$\vec{v}_c = \vec{v}_i - \vec{v}_j - \left(\frac{d_i}{2}\vec{\omega}_i + \frac{d_j}{2}\vec{\omega}_j\right) \times \vec{n}, \qquad (1)$$

where $\vec{v}_i$ and $\vec{\omega}_i$ are the linear and angular velocities before collision of particle i. $\vec{v}_c$ has the normal component $\vec{v}_c^{(n)} = \vec{n}(\vec{v}_c \cdot \vec{n})$ and the tangential component $\vec{v}_c^{(t)} = \vec{v}_c - \vec{v}_c^{(n)}$, so that the vector $\vec{v}_c^{(t)}$ defines the direction of the tangential unit vector $\vec{t} = \vec{v}_c^{(t)}/|\vec{v}_c^{(t)}|$. The impact angle, γ, is defined as the angle between $\vec{n}$ and $\vec{v}_c$ and we always have $\pi/2 < \gamma \leq \pi$.

With the momentum conservation law we define the change of linear momentum of particle i,

$$\Delta\vec{P}_i = m_i(\vec{u}_i - \vec{v}_i) = -m_j(\vec{u}_j - \vec{v}_j) = -\Delta\vec{P}_j, \qquad (2)$$

with $\vec{u}_i$ designating the unknown velocity of particle i after the collision. The normal component, $\Delta\vec{P}_i^{(n)}$, does not affect the angular velocity, whereas the tangential component $\Delta\vec{P}_i^{(t)}$ is connected to a change of angular momentum. $\Delta\vec{P}_i^{(t)}$ is active at the contact point, perpendicular to the moment arm $-(d_i/2)\vec{n}$, so that the vector product of the moment arm and $\Delta\vec{P}_i$ leads to the change of angular momentum

$$-(d_i/2)\vec{n} \times \Delta\vec{P}_i = I_i(\vec{\omega}_i' - \vec{\omega}_i). \qquad (3)$$

In Eq. (3), the moment of inertia about the center of particle i is $I_i = q_i m_i (d_i/2)^2$, and $\vec{\omega}_i'$ is the unknown angular velocity after contact. From $\Delta\vec{P}_i$ and Eqs. 2 and 3 we get the velocities of particle i after the collision without yet knowing $\Delta\vec{P}_i$. The normal component of the momentum change, $\Delta\vec{P}^{(n)}$ can be determined, using the definition of the coefficient of normal restitution $\vec{u}_c^{(n)} = -\epsilon\vec{v}_c^{(n)}$, with $0 \leq \epsilon \leq 1$. This framework holds whether ϵ is velocity dependent or not (Luding et al., 1994a; Luding et al., 1996).

The normal component of the sum $\vec{u}_c - \vec{v}_c = \Delta\vec{P}_i/m_i + \Delta\vec{P}_j/m_j$ leads with the identity $\Delta\vec{P}_i = -\Delta\vec{P}_j$ to the normal component of the momentum change $\Delta\vec{P}_i^{(n)} = -m_{ij}(1+\epsilon)\vec{v}_c^{(n)}$, with the reduced mass $m_{ij} = m_i m_j/(m_i + m_j)$. For the tangential component we assume Coulomb's law, i.e.: $|\Delta\vec{P}_i^{(t)}| \leq \mu|\Delta\vec{P}_i^{(n)}|$, with the coefficient of friction, $\mu \geq 0$. Since the contact is dissipative, the direction of $\Delta\vec{P}_i^{(t)}$ is opposite to $\vec{v}_c^{(t)}$ and thus given by $-\vec{t}$. Therefore, the tangential component of the change of momentum is $\Delta\vec{P}_i^{(t)} \leq \mu m_{ij}(1+\epsilon)v_c \cos\gamma\,\vec{t}$, with $v_c^{(n)} = |\vec{v}_c^{(n)}| = -v_c\cos\gamma$, with $\cos\gamma \leq 0$ for all possible γ. Together with the definition of $\vec{t} = \vec{v}_c^{(t)}/(v_c \sin\gamma)$, we get $\Delta\vec{P}_i \leq m_{ij}\mu(1+\epsilon)\cot\gamma\,\vec{v}_c^{(t)}$. We rewrite this term to get a form that contains the tangential restitution, β, in a form similar to ϵ. Thus, the momentum change is

$$\Delta\vec{P}_i = -m_{ij}(1+\epsilon)\vec{v}_c^{(n)} - \frac{q}{1+q}m_{ij}(1+\beta)\vec{v}_c^{(t)}, \qquad (4)$$

with the prefactor $q = 2/5$ in the moment of inertia for solid spheres. The factor $q/(1+q)$ stems from the fact that a change of linear tangential velocity is always connected to a change of angular momentum. In Eq. (4) we have to restrict momentum change in tangential direction to keep the energy change per collision negative. Therefore, we use $\beta = \min[\beta_0, \beta_1]$. For large γ and thus sticking contacts, we use $\beta = \beta_0$, and for small γ and thus sliding contacts we use $\beta_1 = -1 - ((1+q)/q)\mu(1+\epsilon)\cot\gamma$, following Coulomb's law. This result is consistent with the previously introduced model by (Walton and Braun, 1986) and is also consistent with experimental findings (Foerster et al., 1994).

3 DISSIPATION CUT-OFF

3.1 One particle on a flat plate

First we discuss the case of one particle bouncing on a flat plate with zero horizontal velocity and with a restitution coefficient, ϵ_b, for the particle-bottom collisions. The particle hits the bottom with velocity $v_n > 0$ and bounces back with a smaller velocity $v_n' = -\epsilon_b v_n$. Due to gravity it returns to the plate with velocity $v_{n+1} = -v_n'$. Here the index n indicates the event n, in which the particle is involved. Starting with initial velocity v_0 we have $v_n = \epsilon_b^n v_0$. As an example, the particle has a velocity of $v_{80} = 1.79 \times 10^{-18} v_0$ after 80 collisions if $\epsilon_b = 0.6$. The time between two consecutive events n and $n+1$ is $t_{n+1} = 2v_{n+1}/g$ so that the time until the velocity reaches zero is $t_b = \Sigma_{n=0}^{\infty} t_{n+1} = (2v_0/g)\Sigma_{n=0}^{\infty}\epsilon_b^{n+1} = 2v_0\epsilon_b/[g(1-\epsilon_b)]$. With $v_0 = 1$m/s and $\epsilon_b = 0.6$ one has the time $t_b \approx 0.30581$s until collapse, i.e. divergence of collision frequency, occurs.

Since the ED method handles every collision as one event, a divergence in collision frequency $f_c = 1/t_n$ leads also to a divergence of computing time. Otherwise, every real particle has a certain response time, i.e. every collision takes a certain contact time t_c. Knowing t_c, one may now estimate when the classical ED procedure, that implies $t_c = 0$, gets unphysical in the sense that more than one collisions occur during the contact time t_c, see (Luding et al., 1994a) for examples. Since the restitution coefficient ϵ is defined for pair contacts, it looses its physical meaning if $f_c > 1/t_c$ or equivalently $t_n < t_c$, i.e. if multi-particle contacts occur, or if a particle lies on the bottom.

Therefore, we propose a new definition of the restitution coefficient

$$\epsilon_n^{(i)} = \begin{cases} \epsilon & \text{for } t_n^{(i)} > t_c \\ 1 & \text{for } t_n^{(i)} \le t_c, \end{cases} \quad (5)$$

with the time $t_n^{(i)}$ between two events $n-1$ and n of particle i, and a dissipation cut-off time t_c. This definition is independent of the collision partner of the particle and can be interpreted as some kind of memory of the particle. In other words, the particle performs dissipative collisions if the time between events is larger than the duration of an event t_c. If collisions occur too frequent, the particle becomes elastic. The time t_c, which may be seen as a cut-off time for dissipation, is a material parameter connected e.g. to the elasticity of the material.

If the particle has also tangential motion, we simply set $\beta = -1$ in Eq. 4 if $t_n^{(i)} \le t_c$. This rule together with Eq. 5 conserves the total energy of the system if $t_n^{(i)} \le t_c$ for all $i = 1, ..., N$. The dissipation cut-off in the tangential direction is only one possible rule - others will be discussed elsewhere. The choice for the functional behavior of the transition from a dissipative to an elastic regime is arbitrary. However, since to our knowledge no conclusive theoretical or experimental results exist we use the simple form in Eq. 5.

An ED simulation with the restitution coefficient as defined in Eq. 5 allows one to simulate a particle bouncing on a flat bottom up to times beyond t_b whereas a classical ED simulation with a constant restitution coefficient would have effectively stopped at time t_b. Beyond the transition to the elastic case, the particle bounces with a fixed frequency $f_1 \ge 1/t_c$. Averaging over a time $\tau >> t_c$ one gets the force F_1 exerted by the particle on the bottom as the sum over the momentum changes $2mv_1$ per time τ. The number of colisions in time τ is τf_1 and the velocity at impact is $v_1 = g/(2f_1)$ so that $F_1 = mg$ as expected for a particle lying on the bottom.

3.2 Many particles in a box

Now we discuss situations with N particles in a narrow box, of size $L = 2.001d$, i.e. two particles will settle besides each other in $N/2$ layers. The collisions are rather dissipative, i.e. $\epsilon = \epsilon_w = \epsilon_b = 0.6$ and the particles are smooth, i.e. $\mu = \mu_w = \mu_b = 0$. We let the particles fall using $g = 9.81\text{m/s}^2$ and simulate until time $t = 1$s. We use different numbers of particles in the system, and also vary the cut-off time t_c. In Fig. 1 we plot the number of collisions per particle within time t_c, i.e. $n_c t_c/N$, for different N (left) and different t_c (right). We observe that the total number of collisions per unit time n_c increases as $n_c \propto N^{1+\nu}$ with $\nu > 0$, so that also the computational effort increases with increasing height of the system. Varying t_c leads to an almost constant number of collisions per particle within time t_c, so that the total number of collisions $n_c \propto 1/t_c$.

An estimate for the occurence of the inelastic collapse, see (Luding et al., 1994b) is $n_h(1-\epsilon) > 3$, where $n_h = Nd/L$ is the layer thickness at rest. With this rule, a simulation with $N = 40$ particles, i.e. $n_h(1-\epsilon) = 8$, would run into the inelastic collapse if simulated with a traditional ED algorithm. In Fig. 1 we show that simulations are possible, with the restitution coefficient as proposed in Eq. 5, in a range where inelastic collapse is expected.

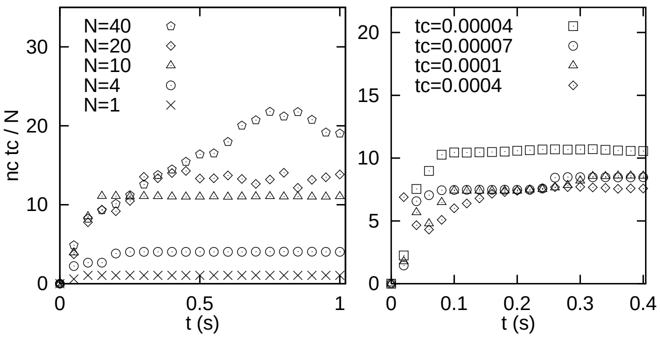

Figure 1. The scaled number of collisions $n_c t_c/N$ versus time, with $t_c = 0.0001s$ and different N (left), and with $N = 10$ and different t_c (right). The paramers are L/d=2.001, ϵ=0.6, and $\mu = 0$.

4 RESULTS

Now we use a vibrating container with the position of the bottom $z_0(t) = A_0 \sin(2\pi f t)$, as a function of time t, and with the amplitude A_0 and the frequency f. In Fig. 2 we plot snapshots from simulations with $t_c = 5.0 \times 10^{-4}$s and with $Nd/L = 14$, i.e. a rather thick layer of particles.

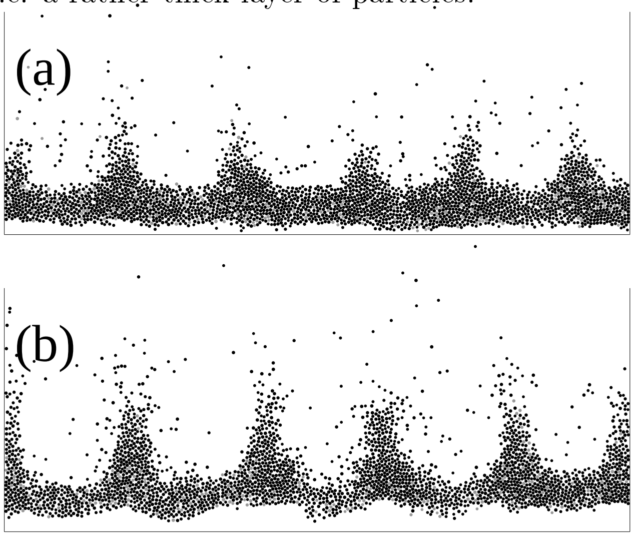

Figure 2. Snapshots from simulations with N=2800 particles in a box of width $L = 200d$. The dimensionless acceleration is $\Gamma = A_0(2\pi f)^2/g = 3.6$ and the frequency is (a) f=11.11 Hz and (b) f=9.524 Hz.

Like (Luding et al., 1996) we use a velocity dependent restitution coefficient $\epsilon(u) = 1 - \epsilon_0(u/u_0)^\gamma$ with $\gamma = 1/5$, $u_0 = 1$m/s, and the relative velocity in normal direction u. We use $\epsilon_0 = 0.4$ for particle-particle and $\epsilon_{0b} = \epsilon_{0w} = 0.2$ for particle-bottom and particle-wall collisions. With increasing velocity the coefficient of restitution decreases with this law. The coefficent of friction is $\mu = 0.2$ and the coefficient of maximum tangential restitution is $\beta_0 = 0.0$. Note that this simulation is not possible with traditional ED simulation since the number of collisions gets too large to be handled in reasonable computing time.

Between two collisions of the array with the vibrating bottom, the pattern can grow, so that a longer period leads to a larger wavelength and amplitude. The standing wave patterns have twice the period of the agitation.

5 DISCUSSION AND CONCLUSIONS

We present here an extension of the classical "hard" sphere ED method that allows to access larger areas in parameter space. In a traditional ED simulation, with high density and strong dissipation, the frequency of collisions may diverge, i.e. the inelastic collapse occurs. The divergence of the number of events per unit time is connected to a total loss of relative energy. We assume elastic collisions if the time between two events gets too small and thus hinder the loss of energy - what in return hinders the divergence of the frequency of events.

We applied this model to the simulation of a large number of particles in a vibrated 2D box and succeed to simulate systems which are not accessible with classical ED. We observe patterns with a period of twice the agitation period and we find that the wavelength and the amplitude of the patterns increases with the period and the time of free flight of the array.

The dissipation cut-off model for the restitution coefficient in normal direction allows one to extend the simulations to higher densities and stronger dissipation. However, the dissipation and the friction in tangential direction are not treated with this approach. A particle that transfers from the dissipative to the elastic state (as defined in this study) has also to change its behavior in tangential direction. We implemented the simplest dissipation cut-off rule in tangential direction, however, more studies are neccessary to elaborate the consequences of different tangential rules.

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