Macroscopically identical granular systems with different number of particles.

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Abstract – One defining property of granular materials is their low number of constituents when compared to molecular systems. This implies that (statistical) fluctuations can have a dominant effect on the global dynamics of the system. In the following letter we create identical macroscopic states with significantly different numbers of particles in order to directly study the role of fluctuations in granular systems. The dependency of the hydrodynamic conservation equations on the particles' size is derived, which directly relates to the total number of particles. We show that, provided that the particles' dissipation is properly scaled, equivalent states can be obtained in the small particle size limit. Simulations of the granular Leidenfrost state confirm the validity of the scalings, and allow us to study the effects of fluctuations on collective oscillations.

Introduction. – Granular flows often show a remarkable similarity with those of molecular fluids [1, 2]. The success of granular hydrodynamic theories in predicting many complex granular behaviours indicates that such a relation is not only superficial [3–9]. But despite continued development, the defining properties of granular materials, such as the dissipative nature of the particles' interactions, still present a challenge for continuum theories, specially for high packing densities or strong dissipations [10–12]. An additional fundamental difficulty stems from the enormous difference in the total number of constituents between granular and molecular systems; while in molecular media the microscopic relevant length-scale is orders of magnitude smaller than the macroscopic one, in granular media macroscopic fields may vary in distances of the order of a few particle diameters. This possibly big influence of a few particles implies the existence of inherently large fluctuations, which can drastically modify the global dynamics, especially near transitions [13–15]. Deepening our understanding of the role played by these fluctuations is thus of fundamental importance for the development of a successful continuum description of granular media.

In the following letter we analyse the influence that finite-number-driven fluctuations have on the macroscopic behaviours of granular matter. For this, we study the possibility of constructing macroscopically identical granular systems with significantly different number of particles. Starting from a given macroscopic hydrodynamic state, and using physical arguments and expressions for the transport coefficients of granular hydrodynamic theory, we derive the dependency on particle size of all terms of the conservation laws. As the particle size is directly related to the total number of particles, we essentially see the dependency of the macroscopic states on the total number of particles present in the system. We demonstrate that in general the granular hydrodynamic equations are not particle-size invariant. Nevertheless, we show that, by properly scaling the restitution coefficient, the limit of vanishing particle size becomes well defined and leads to invariant conservation equations.

The obtained scaling relations are verified by hard-sphere simulations of the granular Leidenfrost state [7, 16, 17]. By computing the coarse-grained fields of equivalent systems we are able to observe the influence of finite-number effects in the macroscopic scale. Fluctuations are seen to have a determinant effect on the oscillatory behaviour previously observed in the same setup although, perhaps surprisingly, they are not seen to affect the characteristic frequency of these oscillations.

Granular hydrodynamic scaling relations. – In the following we study the particle-size dependency of the three-dimensional granular hydrodynamic equations. Our goal is to recreate equivalent hydrodynamic states with significantly different total numbers of particles N. Two hydrodynamic states are considered identical if the invariant macroscopic hydrodynamic fields are the same in both space and time. As invariant hydrodynamic fields, functions of the spatial coordinates $\vec{x} = (x, y, z)$ and time t, we consider the packing fraction, $\phi(\vec{x}, t) = mn(\vec{x}, t)/\rho_p$, with m, d and $\rho_p = 3m/4\pi(d/2)^3$ the mass, diameter and density of the particles, and $n(\vec{x}, t)$ the number density field; the velocity field $\vec{u}(\vec{x}, t) = \langle \vec{v}_i \rangle_{\vec{x},t}$, where $\vec{v}_i$ are the particle velocities; and the fluctuations in velocity, $T^*(\vec{x}, t) \equiv k_B T(\vec{x}, t)/m = \frac{1}{3}(\langle \vec{v}_i^2 \rangle_{\vec{x},t} - \langle \vec{v}_i \rangle_{\vec{x},t}^2)$, with $T(\vec{x}, t)$ the granular temperature field, k_B the granular equivalent of the Boltzmann constant [1], and spatio-temporal averages denoted by $\langle \rangle_{\vec{x},t}$. Physically, in invariant systems we would observe the same packing fraction, velocity and granular temperature distributions in space and time, independent on the number of particles.

In terms of these macroscopic fields the granular hydrodynamic equations can be written as:

$$D_t\phi = -\phi\nabla\cdot\vec{u}, \tag{1a}$$

$$\rho_p\phi D_t\vec{u} = -\nabla p + 2\mu\nabla\cdot\hat{\sigma} + \nabla(\lambda\nabla\cdot\vec{u}) + \rho_p\phi\vec{g}, \tag{1b}$$

$$\tfrac{3}{2}\rho_p\phi D_t T^* = -p(\nabla\cdot\vec{u}) + \lambda(\nabla\cdot\vec{u})^2 + 2\mu\hat{\sigma}:\nabla\vec{u} + \nabla\cdot(\kappa\nabla T^*) + \nabla\cdot(\eta\nabla\phi) - I. \tag{1c}$$

where $D_t = \partial_t + \vec{u}\cdot\nabla$ is the material derivative, $\mathbb{I}$ the identity matrix and $\hat{\sigma} = \frac{1}{2}(\nabla\vec{u} + \nabla\vec{u}^T) - \frac{1}{3}\nabla\cdot\vec{u}\,\mathbb{I}$. They correspond to the mass conservation equation; the momentum conservation equation, with p the pressure, μ the shear viscosity, λ the second viscosity and $\vec{g}$ the acceleration of gravity; and the granular temperature equation, i.e. energy balance, with κ the coefficient of thermal conductivity (note that Fourier's heat law has been used), η an additional transport coefficient present in granular media, and $-I$ the sink of energy-density.

In order for Eqs. (1) to be d-invariant all terms of any equation should scale equally with d. In what follows we determine these scaling relations, starting with heuristic arguments and, after that, using a specific closure for the transport coefficients and the sink term from the literature. First, note that the continuity equation is manifestly invariant, since it only contains hydrodynamic fields and its derivatives, which are by definition identical in equivalent systems. Similarly, the left-hand terms of (1b) and (1c), as well as the gravitational term in (1b), are also d-invariant, if we assume that the particles are made of the same material, i.e., that their density ρ_p is independent of their size[2]. The pressure is proportional to the ideal gas law ($p = \rho_p\phi T^*$) multiplied by a function of the packing fraction only, and therefore the terms in (1b) and (1c) involving the pressure are also d-invariant.

The dependency of the transport coefficients on the size of the particles can be deduced from physical arguments. Transport of mass, momentum or energy from one fluid element into a neighbouring one happens in a layer that has a thickness scaling with the mean free path, l_{fp}. Since the packing fraction ϕ is invariant, l_{fp} should be proportional to the particle diameter d. Therefore all terms in (1b) and (1c) involving the transport coefficients μ, λ, κ and η are expected to scale as d^1.

In contrast, the dissipation term I can be expressed as the number density squared (scaling as d^{-6}) times the energy loss per collision ($\sim d^3$) times the cross-sectional area ($\sim d^2$), leading to $I \sim d^{-1}$. Physically this stands to reason: when the particle size decreases, the growth of the number of collisions is faster than the shrinking of the typical energy loss per collision, and therefore the dissipation per unit volume increases.

Using square brackets, $[\,x\,]_d$, to denote the scaling of a quantity x with d, the above discussion is summarized as

$$[\,p\,]_d = d^0, \tag{2a}$$

$$[\,\mu\,]_d = [\,\lambda\,]_d = [\,\kappa\,]_d = [\,\eta\,]_d = d^1, \tag{2b}$$

$$[\,I\,]_d = d^{-1}. \tag{2c}$$

The immediate conclusion is that the hydrodynamic equations (1b) and (1c) are *not* d-invariant, since they contain terms that scale differently with particle size. In fact, when the particle size decreases, all transport terms decrease, whereas the sink term increases. In general, it is therefore not possible to create identical hydrodynamic states with (significantly) different numbers of particles.

Even if we consider a steady state ($\partial_t = 0$) without macroscopic flow ($\vec{u} = 0$), in which case equations (1a), (1b) and (1c) reduce to

$$\nabla p = \rho_p\phi\vec{g}, \tag{3a}$$

$$\nabla\cdot(\kappa\nabla T^*) + \nabla\cdot(\eta\nabla\phi) = I, \tag{3b}$$

the first equation (3a) is d-invariant, but the second (3b) still isn't. However, for small d invariance can be obtained to order $\mathcal{O}(d^2)$. Both I and η are proportional to the inelasticity $\varepsilon \equiv (1-r^2)$, where r is the coefficient of normal restitution. If we now let r depend on the particle diameter d such that

$$[\,\varepsilon\,]_d = d^2, \tag{4}$$

the $d \to 0$ limit becomes well defined, and $[\,\eta\,]_d = d^3$ (as will be shown next), which for small d (higher N) make the set of equations (3) becomes d-invariant to second order in d.

If the condition (4) is used in the general flow case then,

[1] I.e., the constant that relates the granular temperature scale to the kinetic energy per particle.

[2] In principle, one could let ρ_p scale in any way with the particle size, but this would lead to the same conclusions, only complicating the algebra.

for small d, equations (1) would result in

$$D_t\phi = -\phi\nabla\cdot\vec{u}, \tag{5a}$$
$$\rho_p\phi D_t\vec{u} = -\nabla p + \rho_p\phi\vec{g} + \mathcal{O}(d), \tag{5b}$$
$$\rho_p\phi D_t T^* = -p(\nabla\cdot\vec{u}) + \mathcal{O}(d). \tag{5c}$$

Notice that in the limit $d \to 0$, the above equations converge to those of a perfect (non-dissipative) fluid. Nevertheless, it is important to remark that solutions to these equations are not expected to coincide with solutions of eqs. (1) in the $d \to 0$ limit. For example, it is clear that the absence of dissipation in eqs. (5) will fail to capture the steady state of any driven granular system, as energy would continue to increase indefinitely. Convergence can thus be expected only for non-driven system, where the steady state corresponds to all material being motionless.

Direct derivation of the scaling. – We will now explicitly derive the dependency of the transport coefficients on d considering the expressions obtained by Garzó and Dufty using the Chapman-Enskog method to solve the Boltzmann kinetic equation of the Revised Enskog Theory [18,19]. When expanded using Sonine polynomials, the coefficients take the same general form $\chi = \chi_0\tilde{\chi}(\phi, r)$, where χ_0 are the values for the low-density and elastic limit [19]. The corrections for excluded-volume and finite dissipation $\tilde{\chi}(\phi, r)$ are quite involved; here we are just interested in the fact that they depend only on the packing fraction ϕ and the restitution coefficient r. Therefore, if we assume that r is constant, $[\,r\,]_d = d^0$, then $[\,\tilde{\chi}(\phi, r)\,]_d = d^0$; further along we will elaborate on the consequences of relaxing this condition.

The low-density and elastic limit expressions of all transport coefficients can be expressed as a function of μ_0, as $\kappa_0 = (15/4)\mu_0$, $\eta_0 = (T^*/\phi)\kappa_0$ and, from Stokes approximation, $\lambda = -(2/3)\mu$ [19]. Their explicit form is given by $\mu_0 = (5/96)\rho_p d\sqrt{\pi T^*}$, thus confirming (2b). The energy-density dissipation term follows a similar expression, $I = \frac{144}{5\sqrt{\pi}}\frac{\rho_p}{d}\phi^2 T^{*3/2}\tilde{I}(\phi, T^*, d, r)$, which leads to (2c). Therefore we see that the derived scalings agree with our physical argumentation based on the mean free path.

The functions $\tilde{\chi}(\phi, r)$ can be expanded in terms of ε, which yields

$$\tilde{\mu} = A_0^\mu(\phi) + A_1^\mu(\phi)\varepsilon + O(\varepsilon^2) = -\tfrac{3}{2}\tilde{\lambda} \tag{6a}$$
$$\tilde{\kappa} = A_0^\kappa(\phi) + A_1^\kappa(\phi)\varepsilon + O(\varepsilon^2), \tag{6b}$$
$$\tilde{\eta} = A_1^\eta(\phi)\varepsilon + O(\varepsilon^2), \tag{6c}$$
$$\tilde{I} = A_0^I(\phi, T^*, d)\varepsilon - A_1^I(\phi, T^*, d)\varepsilon^2 + O(\varepsilon^3). \tag{6d}$$

with the coefficients depending only on the shown quantities. We then see that for small dissipations, that is, to leading order in eqs.(6), $[\,\mu\,]_d$, $[\,\lambda\,]_d$ and $[\,\kappa\,]_d$ remain unchanged, while $[\,\eta\,]_d = d\,[\,\varepsilon\,]_d$ and $[\,I\,]_d = d^{-1}\,[\,\varepsilon\,]_d$. If then the dissipation is scaled as eq. (4), I does no longer diverge when $d \to 0$, as $[\,I\,]_d = d$, and η decreases faster than the other transport coefficients, $[\,\eta\,]_d = d^3$.

Having explored the possibility of obtaining macroscopically identical systems with different d (or, equivalently, different N) we now consider a specific granular steady state in which the similarity of equivalent systems can be tested.

Granular Leidenfrost state. – As a test case we consider the granular Leidenfrost state, which consists of a density inverted particle arrangement where a high temperature, gaseous region near a vibrating bottom wall sustains a denser, colder bed of grains on top [16, 17]. As the granular Leidenfrost state corresponds to a steady state with no flow, it is expected to be described by Eqs. (3) (with appropriate excluded volume corrections) [7, 16]. If the energy input is increased, the bed of grains goes through a transition to a buoyancy-driven convective state [15, 20]. Here we avoid this transition by considering a quasi-one-dimensional geometry with base dimensions $l_x = l_y = l = 5d \ll h$, with h the height of the granular column, thus preventing through geometrical constraints the development of convection [21].

Boundary conditions. In order for the macroscopic system to be completely defined only the set of boundary conditions rests to be determined. In steady states the injected energy is equal to the total dissipated energy. To account for the energy injection through particle-bottom wall collisions (3b) is integrated over the whole domain, resulting in

$$J_{\text{in}} = \int_0^h I dz, \tag{7}$$

where J_{in} the energy-density flux injected to the system through particle collisions with the bottom boundary. We have considered that at the free top boundary $J(z) \to 0$ as $z \to \infty$. In order to simplify our system we consider periodic boundary conditions in the lateral directions, and thus only the bottom wall injects energy. J_{in} can be analytically determined in the dilute limit for a sinusoidal driving with $u_b \ll \sqrt{T(z=0)}$ [22,23], expected to be valid in the Leidenfrost state, leading to

$$J_{\text{in}} = \rho_p\phi(z=0)\,u_b^3\left(\frac{2\,T^*(z=0)}{u_b^2}\right)^{1/2} \tag{8}$$

with $u_b = A\omega$ the typical velocity of the bottom plate. Therefore, using our previously derived scalings, and imposing that $[\,h\,]_d = d^0$, it is straightforward to see that, by (7), $\left[\,u_b^2\,\right]_d = d$, and thus $[\,A\omega\,]_d = d^{1/2}$. In our case we use $[\,A\,]_d = d$, to make sure that the amplitude remains smaller than the particle size in the limit $d \to 0$. It then follows that

$$[\,\omega\,]_d = d^{-1/2}. \tag{9}$$

The balance of energy (7) also indicates that the relevant number of particles must be considered per unit base

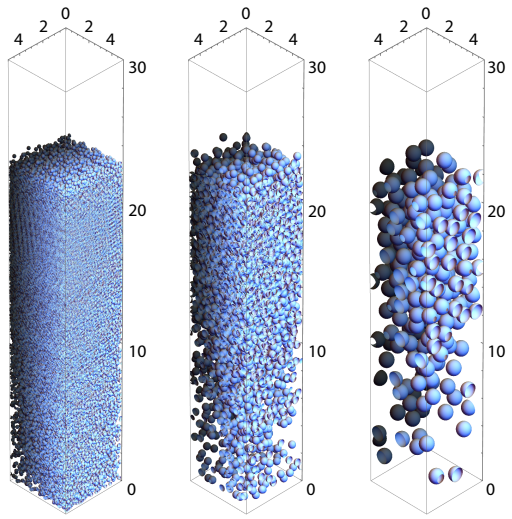

Fig. 1: Configurations of macroscopically equivalent systems in the Leidenfrost state, for particle sizes $d = 0.25$, $d = 0.5$ and $d = 1.0$ from left to right. Notice that to improve the visualization here the base area has not been scaled with d.

area, N/l^2, as in equilibrium the particles in a vertical section dissipate the energy injected by the vibrations of the column's base area. As $N \equiv \int n(\vec{x})\,\mathrm{d}\Omega$, we see that

$$\left[\, N/l^2 \,\right]_d = d^{-3}. \tag{10}$$

In order to reduce the total number of particles in the system and thus decrease the computation time, the dimensions of the container are scaled such that $\left[\, l^2 \,\right]_d = d^2$, Notice that in that case eq. (10) still holds.

Dimensionless quantities. We now derive how do the relevant dimensionless numbers for the Leidenfrost system scale with d. The amount of particles is correctly quantified independent of the system's size by the number of filling layers $F \equiv Nd^2/l^2$. From (10) it is clear that $[\,F\,]_d = d^{-1}$. The shaking strength $S_f \equiv A^2\omega^2/gd$ is known to be a good control parameter for the transition to buoyancy-driven convection [17]. It follows from (9) that $[\,S_f\,]_d = d^0$, implying that the critical points should be d-invariant. Finally, the Reynolds number quantifies the relative importance of inertial to viscous forces, $\mathrm{Re} = \rho_p \phi \vec{v} h/\mu$. Using (2b), we see that $[\,\mathrm{Re}\,]_d = d^{-1}$. The divergence for $d \to 0$ is expected, as we have seen that in that limit the fluid posseses no viscosity.

Simulations. – Numerical simulations are performed using the event-driven discrete particle method [24]. For details about the algorithm's characteristics we refer the reader to [21]. As in the theoretical model we assume collisions are determined by a single coefficient of restitution r. Different systems are referred to as $\mathbb{S}_d \equiv \{d; N_d, l_d, r_d, \omega_d, A_d\}$, with the variables' subscripts denoting the specific d. In order to produce equivalent systems a reference one must be defined, which we take to be $\mathbb{S}_1$,

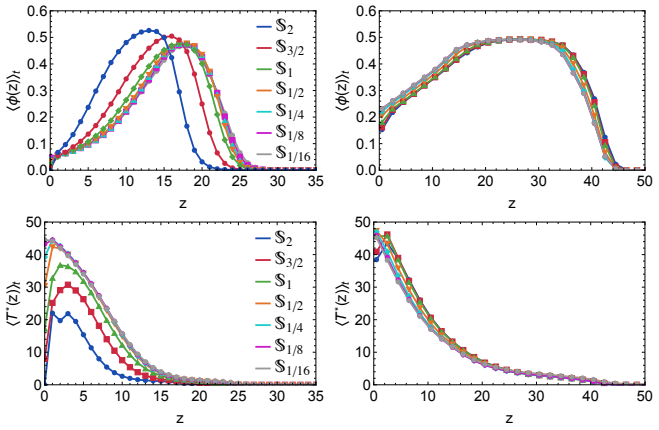

Fig. 2: Time-averaged vertical packing fraction profiles $\langle \phi(z) \rangle_t$ and fluctuating velocity profiles $\langle T^*(z) \rangle_t$, for systems shown in the dissipative ($\varepsilon_1 = 0.9$, left) and quasi-elastic ($\varepsilon_1 = 0.99$, right) cases.

and use $d_1 = 1$ as length-scale for all systems. Time is measured with respect to the acceleration of gravity, in units of $\sqrt{d_1/g}$.

Following the theoretical analysis, we scale the amplitude proportional to d, $A = 0.1d$, and thus $A_1 = 0.1$. The small prefactor is chosen so as to minimize the spatial inhomogeneities induced by higher oscillation amplitudes, and thus approach the limit of an effective fixed temperature boundary condition [25]. The frequency of oscillation is taken such that the system is well into the Leidenfrost state, $\omega_1 = 30\sqrt{g/d_1}$. Finally, two reference particle-particle coefficients of restitution will be considered, $r_1 = 0.9$ and $r_1^e = 0.99$, referred to as dissipative and quasi-elastic systems, respectively (the superscript denotes quantities for the quasi-elastic case). As we want to compare systems with similar packing fractions, in the quasi-elastic case we consider a larger number of particles, so that $F_1 = 12$ and $F_1^e = 32$.

Results. Macroscopic fields are seen to converge in the limit $d \to 0$. Vertical profiles of the time-averaged packing fraction $\langle \phi(z) \rangle_t$ and the fluctuating velocity $\langle T^*(z) \rangle_t$ are shown in fig. 2 for several different $\mathbb{S}_d$. The characteristics of the Leidenfrost state can be readily recognized: low density, high temperature regions near the bottom, below high density, low temperature regions higher up [20]. As expected from (3), only for small d the conserved fields converge, although convergence is fast enough to allow us to fabricate equivalent systems with a difference of more than four orders of magnitude in N/l^2. The gaseous region (close to the bottom boundary) presents the most significant differences, although the maximum of $\langle \phi(z) \rangle_t$ also decreases slightly as $d \to 0$. Variations in the gaseous region are also significant in $\langle T^*(z) \rangle_t$, which presents a twofold increase accompanied by a rising total temperature T_t^* as $d \to 0$.

The shapes of $\langle \phi(z) \rangle_t$ and $\langle T^*(z) \rangle_t$ suggest that an ad-

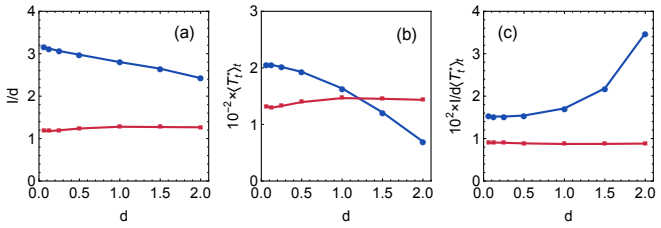

Fig. 3: (a) Total energy-density dissipation per time normalized by d, I_d/d, for dissipative (blue) and quasi-elastic (red) systems from Fig. 2. (b) Total fluctuating velocity T^* for the same systems as in (a). (c) Ratio of the two previous quantities, I_d/dT^*.

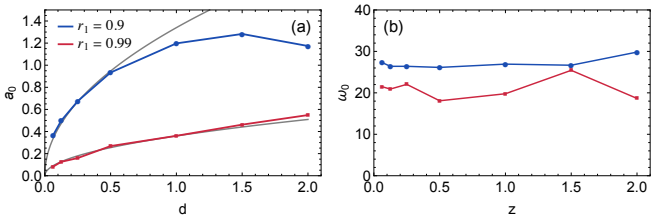

Fig. 4: (a) Amplitude of the low-frequency oscillations a_0, defined as the standard deviation of the centre of mass $\sigma(z_{cm}(t))$, for equivalent $\mathbb{S}_d$. In gray, square root fits. (b) Frequency of oscillation of the column, ω_0, for the same systems as in (a).

ditional source of disagreement comes from finite-size effects. The free-volume near the bottom wall, of course not taken into account in (3), is proportional to d, leading to important disagreements for large d (see fig. 2). Secondly, variations of the convergent macroscopic fields can become comparable to d for large particles; notice for example that in the convergent $\langle\phi(z)\rangle_t$, the height of the gaseous region $h_g \approx 10$, which corresponds to $h_g = 5d_2$.

Beyond finite-size effects, the value of the coefficient of restitution is expected to have a significant influence, as we have taken $\varepsilon \sim d^2$, indirectly affecting all transport coefficients. Indeed, quasi-elastic systems show a much higher agreement as d is varied, as shown in fig. 2 for $\langle\phi^e(z)\rangle_t$ and $\langle T^{*e}(z)\rangle_t$. The most significant differences are again observed near $z = 0$, further suggesting that these are finite-size, boundary-layer effects.

The sources of disagreement can be traced by measuring each term of Eqs. (3) separately. For the energy-density dissipation rate, (3b) states that I scales inversily linear with d, but simulation measurements give a considerable deviation, as shown in fig. 3a. In dissipative systems deviations are of the order of $\sim 20\%$, while in quasi-elastic systems it is only $\sim 5\%$. The improvement for higher r suggests that the deviations stem from neglecting higher order ε dependencies of the transports coefficients.

Deviations from the expected d-invariant behaviour of T^* are even stronger, especially in the dissipative case, as shown in fig. 3b. The quasi-elastic case shows again a significant improvement. Interestingly, for both I/d and T^* the behaviour with d changes between dissipative and quasi-elastic systems: in the former case quantities increase as $d \to 0$, while in the latter they decrease until $d = 1/4$, below which their behaviour cannot be extrapolated by our data. The observed convergence to a macroscopic state, even when the individual scaling relations are seen to deviate, may be explained by the convergence of the general equations (1) to the perfect fluid equations.

Low-frequency oscillations. Shaken beds of grains in density inverted states undergo collective semi-periodic oscillations, referred to as low-frequency oscillations (LFOs) [15, 21, 26]. These are clearly identifiable in all $\mathbb{S}_d$, making it possible to study their properties in equivalent macroscopic systems with different N. Remarkably, their characteristic amplitude, quantified by the standard deviation of the evolution of the vertical centre of mass, $a_0 \equiv \sigma(z_{cm}(t))$, is seen to be proportional to $d^{1/2}$, as shown in fig. 4a. On the other hand, the characteristic frequency ω_0, obtained from the fast Fourier transform of $z_{cm}(t)$, shows a roughly constant behaviour with d, as shown in fig. 4b. The $[\omega_0]_d = d^0$ behaviour is in accordance with a previously derived theoretical expression [21], where disregarding higher order effects, the characteristic frequency was found to be given by $\omega_0^t = g\rho_g/m_s$ with ρ_g the density of the gaseous region and m_s the total mass of the solid region. ω_0^t is thus expected to become d-invariant, as $\phi(z)$ converges for $d \to 0$ and both ρ_g and m_s are macroscopic quantities determined by $\phi(z)$.

From the decrease of a_0 as $d \to 0$ we can conclude that, in the limit of $d \to 0$, LFOs would be unmeasurable, making it an essentially finite-size (granular) phenomena. Moreover, for d small enough, the behaviour of $a_0(d)$ is consistent with a $\sqrt{N}$ law, suggesting that low-frequency oscillations are driven by intrinsic fluctuations due to the low number of particles in the system. We propose the following interpretation: as N decreases, the relative strength of the momentum fluctuations given by particles of the gaseous phase hitting the solid/fluid phase increases, and as such the amplitude of the oscillations are bigger. On the contrary, for smaller d, a significant amount of particles in the gaseous phase would have to transfer momentum to the solid phase at the same time to have an equivalent impact, a situation that becomes increasingly improbable as N increases.

It is interesting to notice that even though the amplitude of LFOs becomes negligible, the mode is still present in the macroscopic system, as the $[\omega_0]_d = d^0$ behaviour shows. This further suggests that LFOs are an intrinsic characteristic of density inverted states, as argued in [21]. The situation is curious, as the mode is a macroscopic phenomena, but its amplitude is driven by microscopic effects. Furthermore, the evolution of $z_{cm}(t)$ is seen to become less chaotic and closer to a harmonic oscillation with a clearly defined frequency as $d \to 0$, as increasingly steep peaks in the Fourier transforms indicate (not shown); this

is another sign that low-frequency oscillations are driven by finite-number fluctuations.

Conclusions. – We have studied the possibility of creating macroscopically equivalent granular systems with significantly different numbers of particles N. Considering the granular hydrodynamic equations, we have demonstrated that it is not possible to obtain equivalent systems in the most general flow case, as different terms scale differently with particle diameter d. Nevertheless, after appropriately scaling the restitution coefficient, the limit $d \to 0$ becomes well defined and leads to a d-invariant set of conservation laws corresponding to those of a perfect fluid. As a consequence of the proper dissipation scaling, the steady-state, fluxless equations become d-invariant in the low-dissipation limit, and for small particle sizes, up to $\mathcal{O}(d^3)$.

Simulations of perfect hard-spheres allowed us to test the derived scalings for a considerable range of total numbers of particles. As a test case we considered the granular Leidenfrost state, which was seen to converge to a limit macroscopic state as $d \to 0$, with the convergence considerably faster for lower energy dissipation. Furthermore, the collective oscillatory behavior (LFO) present in the granular Leidenfrost state was deduced to be driven by the statistical fluctuations in systems with lower numbers of particles. This follows from the decrease of the amplitude of the oscillations with d for macroscopically equivalent systems. Moreover, the frequency of the LFOs remains approximately constant in the range of d studied, in accordance with previously results predicting a dependency only on macroscopic quantities.

As a final comment, we would like to remark that the same framework could be used for the study of other out-of-equilibrium granular states. Macroscopic convergence can be expected for different N, opening the possibility of studying macroscopically equivalent particle systems with significantly different numbers of particles.

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