A comparison of regime-specific continuum models for granular flows

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***Abstract*— Continuum models are efficient mathematical tools used by scientists and practitioners for simulating granular materials under diverse loading conditions. However, material behaviour is determined by constitutive models that typically capture either fluid-like behaviour or solid-like behaviour only, often within limited pressure and strain rate ranges. In this study, three constitutive models, modified Cam-Clay, weakly compressible *μ(I)*-rheology, and a Newtonian fluid are implemented in the Material Point Method (MPM). Their differences are compared in a non-homogeneous, static-dynamic boundary value problem, namely two-dimensional flat and inclined column collapse. The results show that the extreme case of Newtonian fluid cannot support the slope, as expected, while modified Cam-Clay and *μ(I)*-rheology form qualitatively similar heaps, suggesting that unification is a viable option. Modified Cam-Clay captures larger density ranges, but has limited applicability in dynamic flows, while *μ(I)*-rheology persistently flows when expected to stop.**

Keywords: granular materials; constitutive laws; solid-fluid transition; material point method; column collapse

1. Introduction

Grains and powders are fascinating materials that display both solid-like and fluid-like behaviour [1, 2]. Their mechanical behaviour at slow strain rates relies on well-established models. Frameworks such as Critical State Soil Mechanics (CSSM) and Jenike flow theory form an integral part in the design processes of engineering systems (e.g., foundations or silos) [3, 4]. However, at low consolidation stresses and high shear rates, knowledge on the onset and development of flow is limited, especially during the transition from a solid-like to a fluid-like regime.

When considering computational cost, numerical descriptions that reproduce granular materials as continua, triumph over high fidelity particle-level solvers for large systems. However, continuum models are usually regime specific, and hamper the efforts of modelers to capture the full dynamics, states, and transitions, across solid-like and fluid-like regimes.

In this study, we explore the dynamics of three constitutive models: (a) modified Cam-Clay, a solid-like rate-independent model based on the CSSM framework [4], usually adopted to simulate soil; (b) weakly compressible *μ(I)*-rheology, a fluid-like steady-state, rate-dependent model, that captures the steady-state flow behaviour of granular materials; (c) and Newtonian fluid, as simple reference model. These models are implemented into a Material Point Method (MPM) solver to capture the large deformation and material response in boundary value problems, i.e., a column collapse under flat and inclined configurations.

2. Models

2.1. Modified Cam-Clay

The modified Cam-Clay is a precursor to many elastoplastic CSSM models [3]. The yield surface is an ellipse:

$$F \equiv \frac{q^2}{M^2} - p(p_c - p) = 0, \tag{1}$$

where M is the slope of the critical state line (CSL) in the $q-p$ space, with $p = -\left(\frac{1}{\text{dim}}\right)\text{trace}(\boldsymbol{\sigma})$, $q = \sqrt{(3/2)\boldsymbol{s}:\boldsymbol{s}^T}$, $\boldsymbol{\sigma}$ is the stress tensor and $\boldsymbol{s} = \boldsymbol{\sigma} + p\boldsymbol{I}$ its deviatoric part. The pre-consolidation pressure p_c determines the size of the yield surface, and its evolution describes the plastic hardening until critical (steady) state is reached, corresponding to failure. The flow rule is associated, i.e., the yield surface is taken as the plastic potential, therefore the plastic strain increment tensor is defined by:

$$d\boldsymbol{\varepsilon}^p = \frac{\partial \text{F}}{\partial \boldsymbol{\sigma}} = d\Lambda \left[\frac{3}{M^2}\boldsymbol{s} + (2p - p_c)\boldsymbol{I}\right], \tag{2}$$

where $d\Lambda > 0$ is the plastic multiplier, and $\boldsymbol{I}$ is the identity tensor. In the CSSM framework, all virgin material states start from the isotropic compression line (ICL) and fail on the CSL. Those are assumed to be parallel in the $\ln v - \ln p$ space [5], leading to the hardening rule:

$$dp_c = (\lambda - \kappa)^{-1} p_c d\varepsilon_v^p, \tag{3}$$

and the elastic law:

$$dp = \kappa^{-1} p d\varepsilon_v^e. \tag{4}$$

where $d\varepsilon_v^e$ and $d\varepsilon_v^p$ are the elastic and plastic volumetric strain increments, respectively. The parameters κ and λ are the slopes of the ICL and CSL, respectively.

2.2 Newtonian fluid

The total stress tensor for fluid-like models is chosen as

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \eta\dot{\boldsymbol{\varepsilon}}_d, \tag{5}$$

where η is a scalar viscosity, $\dot{\boldsymbol{\varepsilon}}_d$ is the deviatoric strain rate tensors and relates to the scalar shear strain $\dot{\gamma} = \sqrt{(1/2)\dot{\boldsymbol{\varepsilon}}_d : \dot{\boldsymbol{\varepsilon}}_d{}^T}$. Fluid pressure for weakly compressible materials makes use of an equation of state (EOS):

$$p = K\left[\left(\frac{\rho}{\rho_0}\right)^\alpha - 1\right], \tag{6}$$

where α is a compressibility parameter (for water $\alpha = 7$), K is the bulk modulus, ρ and ρ_0 are the initial and current bulk densities.

2.3 Weakly compressible *μ(I)*-rheology

In the case of granular flows, η can no longer be assumed to be constant, but rather dependent on the stress state and strain rate applied. The well-known *μ(I)*-rheology [1, 6] is a model for uniform steady granular shear flows of perfectly rigid, frictionless particles. The dimensionless inertial number relates the pressure state of the flow, and shear strain rate, as

$$I = \frac{\dot{\gamma} d_0}{\sqrt{p/\rho_p}}, \tag{7}$$

where d_0 and ρ_p are the particle diameter and density, respectively. The bulk friction in $q-p$ space is the ratio $\mu = q/\sqrt{3}p$. The relation between bulk friction and inertial number is:

$$\mu(I) = \mu_0 + \frac{\mu_d - \mu_0}{\frac{I_0}{I} + 1}, \tag{8}$$

where I_0 is a characteristic dimensionless inertial constant, μ_0 and μ_d represents the zero and infinite strain rate limits, respectively. A method of determining the viscosity in (7), is by assuming the alignment condition (i.e., shear strain rate and deviatoric stress are co-axial and their eigenvalues carry equal ratios) and a partial regularization [6,7]:

$$\eta(I,p) = \frac{\mu_0 p}{\sqrt{\dot{\gamma}^2 + r^2}} + \frac{\mu_d - \mu_0}{\frac{I_0}{I} + 1}\frac{p}{\dot{\gamma}}, \qquad \dot{\gamma} > 0 \qquad (9)$$

where $r = 0.0001$ is a parameter used to regularize the small $\dot{\gamma}$ and I divergences. We solve pressure using (6) with α=1 (See Ref [8] for details).

3. Method

An Affine Particle in Cell (APIC) explicit MPM using cubic splines is employed. A frozen pre-release version of the (developing) code is openly made available to the public [9]. Two setups are studied. In the first case study, a granular column at a low stress state is allowed to collapse under gravity with the acceleration $g = 9.8\ \mathrm{m.s^{-2}}$. Next, an identical setup, but the frame tilted to an 35° angle enforces more dynamic flows. In both cases, we assume plane strain conditions.

The initial column height and width are 0.4 and 0.2 m, respectively. A constant time step $dt = 3 \times 10^{-6}$ s and a cell size of $h = 0.00625$ m with 4 material points per cell are used. The domain height and width are 0.56 and 1.2 m, respectively. The floor is set to no slip and walls to perfect slip boundary condition. The material is assigned a uniform initial density of $1300\ \mathrm{kg.m^{-3}}$. The parameters of the three models in Section 2 are chosen arbitrarily for demonstration purposes as (a) modified Cam-Clay: $M = 0.6614$, $\lambda = 0.0186$, $\kappa = 0.0010$, and $\nu = 0.3$; (b) weakly compressible $\mu(I)$-rheology: $\mu_0 = 0.3819$, $\mu_d = 0.5718$, $I_0 = 0.279$, $K = 26000\ \mathrm{N.m^{-2}}$, $d = 0.0053$ m and $\rho_p = 2000\ \mathrm{kg.m^{-3}}$; (c) Newtonian fluid with K=26000 $\mathrm{N.m^{-2}}$ and η=0.002 Pa.s.

4. Results

Fig. 1 shows a snapshot of the subsequent heaps after the collapse with the three models in Section 2. The modified Cam-Clay and *μ(I)*-rheology are similar, with the latter showing a slightly longer runout distance, whereas the Newtonian model shows fluid-like behaviour, not supporting a finite slope. The *μ(I)*-rheology predicts a smaller shear stress within the bulk, compared to the modified Cam-Clay. The simulation was stopped at 1.2 s. Modified Cam-Clay showed a (decreasing) kinetic energy of 5.8×10^{-5} J, while *μ(I)*-rheology continue to flow slowly, with a kinetic energy of 2.2×10^{-4} J.

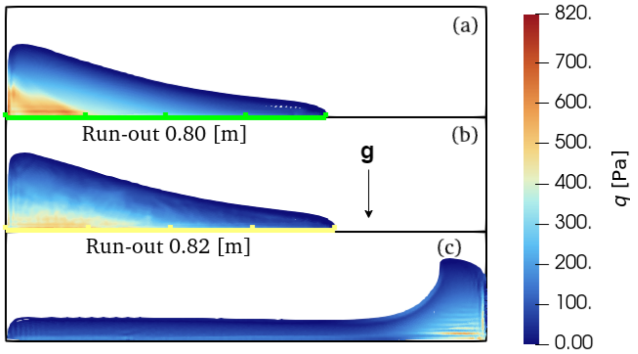

Figure 1. Flat column collapse of three models (a) modified Cam-Clay, (b) weakly compressible *μ(I)*-rheology and (c) Newton fluid. Colour bar represents von-Mises shear stress. Green and yellow lines are rulers. Note, domain boxes are clipped for visibility purposes.

The simulations in Fig. 2 reveal that the modified Cam-Clay, predicts larger density peaks than the $\mu(I)$-rheology. However, the modified Cam-Clay (designed for finite confining stresses) suffers from instabilities as the bulk density decreases and the pressure approaches zero, as indicated by some missing material points in Fig. 2 (a), that take undefined values. On the contrary, the $\mu(I)$-rheology flows and reaches the boundary. The Newtonian fluid flows much faster and further than the granular flow and reached the wall much earlier due to overall lower viscosity. The details and origins for these differences will be discussed in more details elsewhere [10].

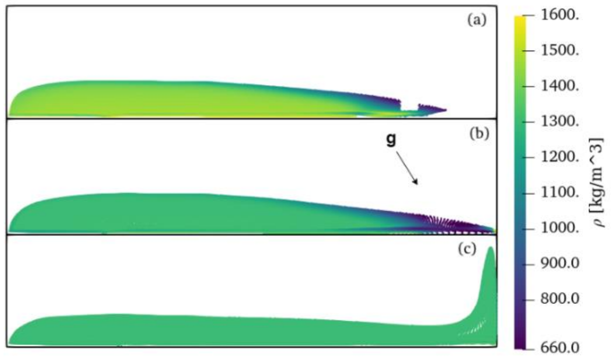

Figure 2. Inclined column collapse of three models (a) modified Cam-Clay, (b) weakly compressible *μ(I)*-rheology and (c) Newton fluid. Domain boxes are clipped for visibility. The viewpoint is rotated at a 35° angle.

5. Conclusion

Overall, this study strengthens the idea that the modified Cam-Clay model and weakly compressible *μ(I)*-rheology are suitable to reproduce granular flows, though in different pressure and strain regimes. Either model has problems in one or the other extreme case. Therefore, a multi-regime model could unify both models, capturing compressible, quasi-static deformations as well as high/low strain rate dynamics and low/large pressures. Interesting features observed from DEM simulations of granular flows with soft particles [2] could potentially be captured by a unified model.

Acknowledgement

This research is part of the project TUSAIL (Training in Upscaling Particle Systems: Advancing Industry across Length-scales, https://tusail.eu) and has received funding from the European Horizon2020 Framework Programme for research, technological development and demonstration under grant agreement ID 955661.

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