Jamming/un-jamming, the transition between solid- and fluid-like behavior in granular matter, is a ubiquitous phenomenon in need of a sound understanding. As argued here, in addition to the usual un-jamming by vanishing pressure and decrease of density, there is also yield (plastic rearrangements and un-jamming that occur) if, for given pressure, the shear stress becomes too large or the density too small. Similar to the van der Waals transition between vapor and water, or the critical current in superconductors, we believe that one mechanism causing yield is by the loss of the energy's convexity. We focus on this mechanism in the context of a simplified version of granular solid hydrodynamics (GSH). Even though any other energy-based theory would display similar transitions, only if it would cover granular gas, fluid, and solid states simultaneously could it follow the system's evolution into un-jammed, possibly dynamic/collisional states – and back to elastically stable ones. We show how the un-jamming dynamics start off and unfolds, and propose an approximation scheme to further simplify its account. It is then employed for illustration, to qualitatively follow the system through various deformation modes: transitions, yielding, un-jamming and jamming, both analytically and numerically.
Un-jamming due to energetic instability: statics to dynamics

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Abstract Jamming/un-jamming, the transition between solid- and fluid-like behavior in granular matter, is a ubiquitous phenomenon in need of a sound understanding. As argued here, in addition to the usual un-jamming by vanishing pressure and decrease of density, there is also yield (plastic rearrangements and un-jamming that occur) if, for given pressure, the shear stress becomes too large or the density too small. Similar to the van der Waals transition between vapor and water, or the critical current in superconductors, we believe that one mechanism causing yield is by the loss of the energy’s convexity.

We focus on this mechanism in the context of a simplified version of granular solid hydrodynamics (GSH). Even though any other energy-based theory would display similar transitions, only if it would cover granular gas, fluid, and solid states simultaneously could it follow the systems evolution into un-jammed, possibly dynamic/collisional states – and back to elastically stable ones. We show how the un-jamming dynamics starts off and unfolds, and propose an approximation scheme to further simplify its account. It is then employed for illustration, to qualitatively follow the system through various deformation modes: transitions, yielding, un-jamming and jamming, both analytically and numerically.

Keywords constitutive model · un-jamming · jamming · concave elastic energy · GSH

Dedication: SL Bob was not only an inspiring researcher and colleague for me, he influenced my research on granular matter so much! Also he became a good friend over the 25 years I knew him. I will always remember the great research visits to Duke, but also the time we spent together on many international conferences, like in Cargese or at several Powders & Grains events. His passing away was a shock and leaves a big gap for me.

Dedication: ML It was in the heydays of helium physics when I, playing with some theories, first met Bob, the conscientious and meticulous experimenter, whose results are wise not to doubt, around which you simply wrap your model. But grains were his real calling. Many decades later, I am again busy fitting my pet theory to his data, and that of his group – such as shear jamming. Some things just never change.

1 Introduction

The macroscopic Navier-Stokes equations allow one to describe Newtonian fluids with constant transport coefficients (e.g., viscosity). In many non-Newtonian systems, especially granular matter, the transport coefficients depend on various state variables such as the density and the granular temperature. This interdependence and the presence of energy dissipation is at the origin of many interesting phenomena: clustering, shear-band formation, jamming/un-jamming, shear-thickening or shear-jamming, plastic deformations, related also to creep/relaxation, and many others; see the chronologically sorted references (which are cited below, where relevant): [12][13][14][15][16][17][18][19][20][21][22][23][24][25][26][27][28][29][30][31][32][33][34][35][36][37][38][39][40][41][42][43][44][45][46][47][48][49][50][51][52][53][54][55][56][57][58][59][60][61][62][63][64][65]
1.1 About states of granular matter

When exposed to external stresses, grains are elastically deformed at their contacts. In static situations, there is only elastic energy; in flowing states, some of the elastic energy is transferred to the kinetic one and back.\footnote{Flowing states, as defined here, range from dilute granular gases via inertial, collisional granular fluids, to quasi-static flows, granular solids, e.g., perturbed by elastic waves, excluding only static, elastic solids. Granular solids and quasi-static flows show both solid and fluid features\cite{42}, in particular a considerable permanent elastic energy. The ratio of kinetic to potential, elastic energy in the system, $K = E_{\text{kin}}/E_{\text{pot}}$, is one way to characterize its state: gas ($K \gg 1$), dense collisional flow ($K \sim 1$), quasi-static flow ($K \ll 1$), granular solid ($K \approx 0$), static ($K = 0$) and the extreme, athermal case ($K \equiv 0$, maintained at all times), as can be realized by energy minimization, e.g., see Ref.\cite{82} and references therein. The contribution of potential energy to the total energy is thus $1/(1 + K)$, and the fraction of the total energy that is exchanged between the kinetic and potential energy is then: $\text{gas} (w_T/w = 2/(1 + K) \ll 1)$, collisional $(w_T/w \sim 1)$, quasi-static and solid $(w_T/w = 2K/(1 + K) \approx 2K \ll 1)$.}

The capability of granular solids to remain quiescent, in mechanical equilibrium, under a given finite stress is precarious. If pressure or shear stress become too large, the grains will, suddenly, start moving – with a vanishing elastic stress. This qualitative change in behavior is an unambiguous phase transition. We shall refer to the region capable of maintaining the equilibrium of static grains as elastic, and its boundary (in the space spanned by the state variables) as the yield surface.

Granular systems will also un-jam for vanishing pressure and a continuous reduction of density, though we reserve the term yield for the (sudden) loss of elastic stability: Grains un-jam in either case, they yield only when the elastic stress, in particular the pressure, is finite.

Starting from the elastic region, decompression (ten
d\)sion) reduces the density and the elastic deformations of the grains – until the latter vanish and the system un-jams. Decompressing further just reduces the density accordingly. The system is now un-jammed in the sense that one can change the density without any restoring force, i.e., the elastic energy remains zero. In reverse, compression only increases the density, as long as it is smaller than the jamming density. At jamming both the elastic deformations and the associated energy start to increase with density.

In contrast, there is a discontinuity leaving the elastic regime at finite values of elastic stress. It is a sudden transition from quiescent, enduringly deformed grains to moving ones oscillatorily deformed. This transition needs to be explained, to have a model for. And it is clear that the transition must be encoded in the elastic

Some open questions are: How can we understand those phenomena that originate from the particle- or meso-scale, which is intermediate between atoms and the macroscopic, hydrodynamic scale? And how can we formulate a theoretical framework that takes the place of the Navier-Stokes equations?

A universal theory must involve all states granular matter can take, i.e., granular gases, fluids, and solids, as well as the transitions between those states. What are the state variables needed for such a theory? And what are the parameters (that we call transport coefficients) and how do they depend on the state variables?

Main goal of this paper is to propose a minimalist candidate for such an universal theory, able to capture granular solid, fluid, and gas, as well as various modes of transitions between these states. The model, remarkably, involves only four state variables, density, momentum density (vector), elastic strain (tensor), and granular temperature. It is a boiled down, simplified case of the more complete theory GSH\cite{60,67,68,69,70,71}. For the sake of transparency and treatability, we also reduce most transport coefficients and parameters to constants – without loss of generality.

Each transport coefficient is related to the propagation or evolution of one (or more) of these quantities that encompass the present state of the system. For simple fluids\cite{3,72}, it is possible to bridge between the (macroscopic) hydrodynamic and the (microscopic) atomistic scales; as an example, the diffusion coefficient quantifies mass-transport mediated by microscopic fluctuations.

In the case of low density gases, the macroscopic equations and the transport coefficients can be obtained using the Boltzmann kinetic equation as a starting point. For moderate densities, the Enskog equation provides a good, quite accurate description of dense gases (or fluids) of hard atoms\cite{3} or of particles including the effects of dissipation\cite{10}, reaching out (empirically) towards realistic systems\cite{72}, and beyond, see, e.g.,\cite{38,60}. At the limit of granular fluids, other coefficients, like the viscosity, actually are observed to diverge\cite{42,75} when the granular fluid becomes denser and approaches jamming to the state that we could call a granular solid, as related to the classical solid mechanics\cite{76}. One objective of this paper is to bring together fundamental theoretical concepts of continuum mechanics\cite{77,78,79,80,55} with observations made from particle simulations for simple granular systems in the gas, fluid, and solid states, including also the transitions between those states\cite{73,50,81,82,75,53}.
energy – the only quantity characterizing the quiescent state – not in the (inactive) dynamics.

In the elastic region, grains appear solid when at rest, but they will flow if subject to an imposed shear rate, and appear liquid. This *continuous change in appearance* is well accounted for by any competent dynamic theory or rheology, it is not a transition.

Moreover, flowing grains in the elastic region do sport a macroscopic elastic shear stress, with an associated elastic energy (even though granular contacts switch continually), something no Newtonian liquid is capable of. Also, the shear stress remains finite when the grains stop flowing, which is not the case in Newtonian fluids.

So there are two different flowing states, either with finite elastic stress/strain, or with vanishing ones, which includes granular gases as accounted for by the kinetic theory, see Ref. [73] and references therein. There is also a transition between them. We take both transitions, either leaving the quiescent state, or the flowing one, as the same transition, with the same underlying physics. (In fact, encoding the first transition in the elastic energy certainly affects the flowing state as well.)

We also assume that the elastic energy possesses only a single mechanism for yield, irrespective whether the pressure or the shear stress is too large, or the density too small, as traditionally encompassed by concepts like plastic potentials, yield functions, or flow rules [40, 77, 78], see Fig. 1 below and textbooks like Ref. [77].

### 1.2 Relation to other systems in physics

We do not think that the transition is due to *spontaneously broken translational symmetry* – the usual mechanism giving rise to static shear stresses, as in any fluid-solid transitions. The quick argument is: Consisting of solid, grains already break translational symmetry. More importantly, the loss of equilibrium and granular static is caused by the shear stress or pressure being too strong. This is an indication of an over-tightening phenomenon, of which the (pair-breaking) *critical current* is a prime example.

If a superconductor conducts electricity without dissipation, it is in a *current-carrying equilibrium state*. If, however, the imposed current exceeds a maximal value, the system leaves equilibrium and enters a dissipative, resistive state. The superfluid velocity, \( v_s \approx \nabla \phi \),

\[ j_s = \frac{\partial u}{\partial v_s} \]

given by the gradient of a quantum mechanical phase, is the analogue of the strain. The dissipationless current, \( j_s \), given by the derivative of the energy with respect to \( v_s \), is the analogue of the elastic stress. The over-tightening transition in superconductivity is well accounted for by an inflection point, at which the energy turns from stably convex to concave, see the classic paper by Bardeen [80]. The close analogy between the two systems is a good reason to employ the same approach here, to postulate that the surface of the cone in Fig. 1 is an inflection surface of the elastic energy.

### 1.3 About elastic granular matter

The granular solid state is contingent on granular matter capable of being elastic, for which there is ample evidence, see e.g. Refs. [77, 81, 88, 11, 89, 90, 91, 92, 93] and references therein. In addition to the material stiffness, many other material properties (including cohesion, friction, surface-roughness, particle-shape) determine the elastic response of granular matter. For soft and stiff materials the deformations are, respectively, considerable and slight, but never zero. Because of their Hertz-like non-linear contacts, grains are infinitely soft and dispersion, see e.g. Ref. [91, 92, 93] and references therein. The discreteness and disorder of granular materials displays non-linearity due to their Hertz-type contacts, on-top of the contact network (fabric) and its re-structuring. Only in computer simulations is it possible to remove the first and focus on the second, see e.g. Ref. [53].

Elastic waves propagate in granular media, displaying various non-linear features, including anisotropy and dispersion, see e.g. Ref. [91, 92, 93] and references therein. The discreteness and disorder of granular media add various phenomena – already for tiny amplitudes – such as dispersion, low-pass filtering and attenuation [91, 92, 93]. With increasing amplitudes, a wide spectrum of further phenomena is unleashed, among which the beginning of irreversibility and plasticity, see Ref. [59] in this topical issue, and references therein, and the loss of mechanical stability [96], what we call “yield” in the following.
1.4 Yield: About the limits of elasticity

To envision the yield surface, we consider the space spanned by three parameters: pressure \( P \), shear stress \( \sigma_s \), and void ratio \( e = (1 - \phi)/\phi \) (where \( \rho = \rho_0 \phi \), with material density \( \rho_0 \) and volume fraction \( \phi \)), ignoring the granular temperature (i.e., fluctuations of kinetic energy), as discussed in Ref. [74] and so many papers following. Based on the observation of the Coulomb yield and the virgin consolidation line, we assume that the yield surface is as rendered in Fig. 1. Elastic, jammed states, maintained by deformed grains, are stable and static only inside it.

The Coulomb yield line, see Fig. 1(b), can be reached by increasing the shear stress at given confining pressure. When the shear stress exceeds a certain level, the system yields, un-jams and becomes dynamic. No static, stable elastic state exists above the Coulomb yield line, as evidenced by a sand pile’s steepest slope.

It is imperative to realize that (what we call) the Coulomb yield line is conceptually different from the peak shear stress achieved during the approach to the critical state at much larger strains. Coulomb yield is the collapse of static states – such as when one slowly tilts a plate carrying grains until they start to flow (max. angle of stability). Its behavior is necessarily encoded in the system’s energy, because this phenomenon does not at all involve the system’s dynamics. The critical state, including the peak shear stress – though referred to as “quasi-static” – is a fully dynamic and irreversible effect. It is accounted for by the stationary solution at given strain rates in GSH. The angle of repose (always smaller than the max. angle of stability) is in GSH given by the critical friction angle [70-71].

In the absence of shear stresses, the maximally sustainable pressure depends on the void ratio, \( e \), as rendered in Fig. 1(a). Starting from a given \( e \), slowly increasing \( P \), the grain-structure will collapse and yield at this pressure, to a smaller value of \( e \), such that the final state is stable, static, and below the curve of Fig. 1(a). This is because when applying a slowly increasing pressure, the point of collapse is (ever so) slightly above the curve; and the end point below it is typically also close. This evolution resembles a stair-case, with the granular medium increasing its density by hugging this curve, which frequently referred to as the virgin/primary consolidation line, or simply the pressure yield line. The line cuts the \( e \)-axis at the random loosest void ratio, \( e_0 \), above which no elastic stable states exist.

However, this does not exclude the possibility that there are plastic deformations possible inside (in finite systems) as evidenced from particle simulations, e.g., in Refs. [22,63].

Because of the pressure yield line, the Coulomb yield curve cannot persist for arbitrarily large \( P \) at given \( e \). Rather, it bends over to form a “cap”, as rendered in Fig. 1(b), since an additional shear stress close to the pressure yield line will also cause the packing to collapse. (The shape of the cap depends on the interplay of isotropic and deviatoric deformations as well as the probability for irreversible, possibly large-scale re-structuring events of the micro-structure, or contact network.)

Merging 1(a) and 1(b) yields the elastic region below the yield surface, as given in Fig. 1(c). Although the \( e \)-axis, for \( P, \sigma_s = 0 \), see Fig. 1 is also referred to as the

Fig. 1 Granular yield surface, or the jamming phase diagram, for \( T_q = 0 \), as a function of the pressure \( P \), shear stress \( \sigma_s \), and void ratio \( e \), as rendered by an energy expression in [69]. Panel (c) is the 3D combination of (a) and (b); with (b) depicting how the straight Coulomb yield line bends over, depending on the void ratio \( e \) – a behavior usually accounted for by cap models in elasto-plastic theories; while (a) depicts the maximal void ratio \( e \) (equivalent to the density) plotted against pressure \( P \), or the so-called virgin consolidation line (VCL). In panel (a), the dotted line is an empirical relation, \( e = e_1 - e_2 \ln(P/P_0) \), with \( P_0 = 0.5 \) MPa, \( e_1 = 0.679 \) and \( e_2 = 0.097 \), approximating the VCL, but not valid for \( P \rightarrow 0 \). The thick solid line cuts the \( e \)-axis at \( e_0 \), with the intersection being the lowest possible, random loosest packing value, see Ref. [69] for details, where also the thin solid line is discussed. Thus \( e_0 \) also defines the lowest possible jamming volume fraction, \( \phi_{J0} = 1/(1 + e_0) \), see Ref. [53], with static, elastic states possible only below the VCL, as will be shown in Sects. 5 and 6.
loci of (isotropic) un-jamming, the elastic stress goes
to zero here, because the grains are successively less deformed. There is, as already discussed above, no phase transition or yield here.

Next, we summarize all different symbols and nomenclatures, as reference.

1.5 Notation and symbols

This paper is a cooperation of co-authors, whose notational baggage from past publications clash with one another. In the dire need to compromise, we ask the readers to suffer – with us – using varying symbols and notations. Our state-variables are: density, \( \rho \), momentum density, \( \rho \mathbf{v} \), granular temperature, \( T_g \), and the elastic strain, as summarized here.

1. The bulk density, \( \rho \), is related to the volume fraction, \( \phi = \rho / \rho_p \) (with \( \rho_p \) the particles’ material density), the porosity \( 1 - \phi \), and the void ratio \( e = (1 - \phi)/\phi \). (Later, we shall choose units such that \( \rho_p = 1 \), so that volume fraction and bulk density are identical.)

2. The conserved momentum density \( g_i \) defines the velocity \( \mathbf{v}_i = g_i/\rho \). The symmetric part of the velocity gradient is

\[
v_{ij} := v_{(i,j)} := \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) = -\varepsilon_{ij} = D_{ij}
\]

The total strain rate \( \varepsilon_{ij} \) is positive for compression and negative for tension.

The symbol \( v_{ij} \) is usual in condensed matter physics, see \([72,76,98]\). It is also the one employed in most previous GSH-publications. The notation \( D_{ij} \) is common in theoretical mechanics \([78,54]\), while \( \varepsilon_{ij} \), or \( \gamma \), are used, e.g., in soil mechanics and related literature \([77]\).

3. Subscripts, such as \( i,j,k,l \), refer to components of tensors in the usual index notation, with double-indices implying summation, the comma indicating a partial derivative, as in \( v_{(i,j)} \); the superscript \( ^* \) denotes the respective traceless (deviatoric) tensor.

Using the summation convention, the volumetric strain-rate is abbreviated as:

\[
\varepsilon_v = \varepsilon_{ii} = -v_{tt} = -D_{tt} = -\mathbf{v} \cdot \mathbf{D},
\]

where the last term is in symbolic tensor notation. The deviatoric strain-rate is thus

\[
\varepsilon_{ij} = -v_{ij} = -D_{ij}',
\]

with the norm \( v_s := \sqrt{\varepsilon_{ij} \varepsilon_{ij}} = \sqrt{\varepsilon_v} \), the deviatoric invariant, insensitive to the sign convention.

4. The elastic strain, \( \varepsilon_{ij}^e \equiv -u_{ij}^e \), is the tensorial state variable on which the elastic (potential) energy depends \(^4\). It is always well-defined and unique, in contrast to the total or plastic strains, which are not, and thus will not be used as state variables for (constitutive) modeling. The respective strain rates, however, are well-defined and thus are used. The strain rate was already given (see item 2.), \( \dot{\varepsilon}_{ij} = -v_{ij} \), so that the plastic strain-rate is defined as: \( \dot{\varepsilon}_{ij}^p = \dot{\varepsilon}_{ij} - \frac{d}{dt} \varepsilon_{ij} \) (see also item 7.).

5. The isotropic elastic strain

\[
\Delta := -u_{ll} = \varepsilon_{ll}^e = e_v^e = \ln (\rho/\rho_J)
\]
is positive for compression. It may be seen as the true strain relative to a stress-free reference configuration – if \( \Delta > 0 \). Arriving at \( \Delta = 0 \), the system un-jams and the jamming density \( \rho_J = \rho \) is the actual one \( \rho \).

6. The norm of the deviatoric elastic strain is, in accordance to the general scheme, \( u_s = \sqrt{\varepsilon_{ij}^e \varepsilon_{ij}^e} = (2 J^2)^{1/2} \).

7. In general, we take \( \frac{d}{dt} \equiv \frac{\partial}{\partial t} \) as the partial time derivative, and \( \frac{d}{\partial t} \) as the total one, including all convective terms. Hence, with the vorticity tensor given as \( \Omega_{ij} \equiv v_{(i,j)} = \frac{1}{2} (\nabla_i v_j - \nabla_j v_i) \), one has (as example) the total time derivative of the elastic strain

\[
\frac{d}{dt} \varepsilon_{ij}^e = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \varepsilon_{ij}^e + \Omega_{ik} \varepsilon_{kj}^e - \varepsilon_{ik}^e \Omega_{kj}.
\]

Being off the focus here, the convective terms are usually neglected, so that \( \frac{d}{dt} = \frac{\partial}{\partial t} \). The dots in \( \varepsilon_{ij}^e \) and \( \dot{\varepsilon}_{ij} \) are only a (convention preserving) indication of rates, but do not represent the mathematical operation above.

8. The total stress is not an independent state variable, but rather given by the energy density and entropy production, as discussed in the classical GSH

\(^4\) Note the different signs in the last two terms, i.e., the isotropic elastic strain, \( \Delta = e_v^e \), is positive for compression, whereas \( u_s^e \) is negative (if eigenvalues are considered).

\(^5\) Generalizing GSH, we allow negative elastic strains \( \Delta = e_v^e \) here, interpreting it as the separation distance between particles – or their mean free path – in order to catch both jammed and un-jammed situations. Note that the elastic energy of a negative \( \Delta \) is identically zero, and that a negative \( \Delta \) is not independent of the density \( \rho \). Compressing from an un-jammed state, the system jams at \( \Delta = 0 \), towards \( \Delta > 0 \) and \( \rho > \rho_J \). In isochoric situations (constant density), an evolution of the state variable, \( \Delta \), the isotropic elastic strain, implies an evolution of the (enveloped, dependent) jamming density, \( \rho_J = \rho \exp(-\Delta) \), as proposed and studied in detail in Ref. \([53]\). The physics clearly changes between positive (jammed) and negative (un-jammed) states, but for the sake of brevity, below jamming, we limit \( \rho_J \geq \rho_{J,0} \) and thus \( \Delta = \ln(\rho/\rho_{J,0}) \), in cases where it would drop below its absolute limit, \( \rho_{J,0} \), which can be seen as the random loosest packing density.
The granular temperature used in GSH is $T$ throughout. Only in the atomic/molecular limit of "particles" one has leads to values of order of the inner temperature of the sun. $T$ resembles. Conversely, one may use $T$ granular gases, if thermal equilibrium could ever be reached, for more details see subsection 3.2.1.

$\text{In what follows, we shall, in Sec. 2, consider the significance of an inflection surface, of a convex-concave transition in the energy, as relevant for classical systems, transiently elastic systems and granular matter. We then present review of and a minimalist version of GSH in Sec. 3, allowing for analytic limits in Sec. 4 and numeric calculations in Sec. 5 before we conclude in Sec. 6.}$

Note that (calligraphic) symbols $B \neq B$, $G \neq G$, and $A$, in general, are the (tangent) moduli, representing the second derivatives of the elastic energy density with respect to isotropic and deviatoric strain, or mixed, respectively; symbols $B_\Delta$, $G_\Delta$ are again different and are the secant moduli; for more details see subsection 4.2.1.

$^7$ The two temperatures $T_g$ and $T_G$, in granular solid, becomes equal to the true temperature, $T_g = T$. In granular gases, if thermal equilibrium could ever be reached, we have $T_G = T$ – a relevant condition if one starts to consider the dissipation and heating of the grains. By ignoring $T_G$'s role as a "temperature" of the granular degrees of freedom, taking it only as a measure of the velocity fluctuation squared, $T_G \sim \langle \delta v_i^2 \rangle$, one may go on using $T_G$ in denser ensembles. Conversely, one may use $T_g$ in granular gases, taking it as $T_g \sim \langle \delta v_i \rangle$. However, while $T_g \approx T$ does hold in granular static equilibrium, $T_G = T$ can never be reached, as any finite $T_G$, for finite sized particles, translated into temperature, leads to values of order of the inner temperature of the sun. Only in the atomic/molecular limit of "particles" one has $T_G$ analogous to $k_B T$. It is therefore more sensible to employ $T_g$ throughout.

2 Equilibrium conditions and dissipative terms

In this section, we first revisit the reason for thermodynamic energy’s convexity, and derive the equilibrium conditions for three systems: elastic, transiently elastic and granular media. There is one equilibrium condition for each state variable, that maximizes its contribution to entropy or, equivalently, minimizes its contribution to energy. Examples for equilibrium conditions are uniform temperatures and uniform stresses. As these conditions represent extremal points, the energy needs to be convex to be minimal, for the system to be stable.

Then we make the general point that every equilibrium condition, if not satisfied, is a dissipative channel that gives rise to a negative/dissipative term in the evolution equation of the associated state variable. As a result, the state variable relaxes, towards satisfying the condition. In a closed system, all variables will eventually satisfy all their respective conditions, which is the state we called equilibrium.

If the energy is concave, equilibrium conditions represent maxima of the energy with respect to variation of a state-variable. The dissipative terms will thus drive the system away from equilibrium, producing, e.g., non-uniformity in temperature and stress fields. When this happens, what micro-mechanical mechanisms it originates from, is necessarily more specific. How the dynamics further evolves depends on the system one considers. In the classical van der Waals theory of the gas-liquid transition, droplet formation is the basic mechanism. In granular media, we propose the following mechanism.

In the stable region, within the cone of Fig. 1, the dissipative term in the equation for the elastic strain serves to maintain stress uniformity. It remains inconspicuous as long as one studies the evolution of uniform stresses. Outside the yield surface, it forces the system to leave stress uniformity. Non-uniform stresses accelerate grains in varying directions, producing jiggling and thus granular temperature which, in turn, allows the stress to relax, pushing the system back into the convex region.

This is what we believe happens in grains at yield and beyond the transition. Setting up a dynamical model for following the system through the transition to different states is the main purpose of this paper.

2.1 Elasticity

Consider an elastic system characterized by two state variables, the entropy density, $s$, and the elastic strain,
\[-\varepsilon_{ij}^i \equiv u_{ij} = \frac{1}{2}(\nabla_i U_j + \nabla_j U_i), \tag{3}\]

with a thermodynamic energy density that is a function of both, \(w = w(s, u_{ij})\) \[16\].

A textbook proof of energy convexity considers only the entropy as a variable, and involves an elastic system connected to a heat bath. A temperature fluctuation (associated to entropy fluctuations) vanishes only if the energy is larger with it than without, which is shown to imply convexity \[90\].

In a more general consideration, we start with the assumption that the system is stable and has an equilibrium for given values of \(s\) and \(u_{ij}\). Since the elastic stress, \(\pi_{ij} \equiv -\partial w/\partial u_{ij}\) is symmetric, \(\pi_{ij} = \pi_{ji}\), we may write the total differential of the energy density as:

\[dw = Tds - \pi_{ij}du_{ij} = Tds - \pi_{ij}d\nabla_j U_i, \tag{4}\]

with temperature \(T = \partial w/\partial s\). We varied this energy by (i) keeping \(\int dV = \text{const.}\), or \(\int (w - T_L s) dV = 0\) with \(T_L = \text{const.}\) a Lagrange parameter; (ii) forbidding external work \(\int \pi_{ij}dU_i dA_j = 0\); and (iii) using Gauss' theorem \[7\] the result is

\[0 = \int \left[ T\delta s - \pi_{ij}d\nabla_j U_i - T_L\delta s \right] dV \]
\[= \int \left[ (T - T_L)\delta s + (\nabla_j\pi_{ij})dU_i \right] dV. \tag{5}\]

With \(\delta s\) and \(\delta U_i\) varying independently, and \(T_L = \text{const.}\), the equilibrium conditions may be written as

\[\nabla_i T = 0, \quad \nabla_j \pi_{ij} = 0. \tag{6}\]

These are extremal conditions. They represent an energy minimum and stable equilibrium, only if deviations from them yield an energy increase. Therefore, inserting \(T = T^{eq} + \delta T, \pi_{ij} = \pi^{eq}_{ij} + \delta \pi_{ij}\), with \(\nabla_i T^{eq} = 0\) and \(\nabla_j \pi^{eq}_{ij} = 0\), we require

\[\delta^2 w = \delta T\delta s - \delta \pi_{ij}\delta u_{ij} > 0. \tag{7}\]

Assuming first \(\delta u_{ij} \equiv 0\), we may write \(\delta^2 w = \delta T\delta s = (\partial T/\partial s)(\delta s)^2 > 0\), implying

\[\frac{\partial^2 w}{\partial s^2} = \frac{\partial T}{\partial s} > 0, \tag{8}\]

or that the energy \(w\) is a convex function of \(s\). As a result, temperature fluctuations will diminish, and the state characterized by a uniform temperature is a stable equilibrium. Conversely, if the energy is concave, \(\partial^2 w/\partial s^2 < 0\), the condition \(\nabla_i T = 0\) represents a maximum of energy, and the system is unstable. Any fluctuations in entropy will move it away from uniform temperature. In the case of the van der Waals transition between gas and liquid, a uniform single-phase system is moved to the coexistence of two phases, with different entropy densities, but the same temperature.

Next, as used explicitly below, in subsection 3.2.1 assuming \(\delta s \equiv 0\), we order the six components of \(\pi_{ij}\) and \(u_{ij}\) each as a 6-tuple vector, denoted by Greek letters, and require

\[\delta^2 w = -\delta \pi_{ij}\delta u_{ij} = -\delta \pi_\alpha \delta u_\alpha = \frac{\partial \pi_\alpha}{\partial u_\beta} \delta u_\alpha \delta u_\beta > 0. \tag{9}\]

This implies that the 6x6 Hessian matrix

\[\frac{\partial^2 w_{eq}}{\partial u_\alpha \partial u_\beta} = -\frac{\partial \pi_\alpha}{\partial u_\beta} \]

implies that the energy \(w\) is a convex function of the elastic strain \(u_{ij}\). If there is at least one negative eigenvalue, the condition \(\nabla_j \pi_{ij} = 0\) no longer represents a stable state, because along the associated eigenvector, the energy is a maximum. The system can and will escape, initially by violating \(\nabla_j \pi_{ij} = 0\), typically rendering the stress non-uniform.

To obtain static elastic solutions, we solve \(\nabla_i \pi_{ij} = 0\) for given boundary conditions. This is equivalent to looking for minima of the elastic energy. The solutions are stable if the elastic energy is convex. They are unstable otherwise, and devoid of physical significance.

The more general consideration, including both \(\delta s\) and \(\delta u_{ij}\), leads to a 7x7 matrix that, for stable equilibrium, must possess seven positive eigenvalues.

A complete consideration for elasticity requires also the inclusion of the density, \(\rho\), and momentum density \(\rho v_i\), as the energy’s variables. This, being somewhat more lengthy, would distract from the present concern. The associated equilibrium conditions, with the gravitational acceleration, \(g_i\), and the chemical potential given as \(\mu = \partial w/\partial \rho\) (as derived in Refs. \[100,129\]) are:

\[\nabla_i \mu = -g_i, \tag{10}\]
\[-\varepsilon_{ij} \equiv \text{tr} \equiv \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) = 0, \tag{11}\]
\[\nabla_i P = s\nabla_i T + \rho \nabla_i \mu = -\rho g_i. \tag{12}\]

The force equilibrium \(\nabla_i P = -\rho g_i\) is a direct result of \(\nabla_i T = 0\) and \(\nabla_i \mu = -g_i\). All three equations express minimal energy, or maximal entropy.
If any of the equilibrium conditions are not satisfied, dissipative currents appear to counteract: heat diffusion $\sim \nabla^2 T$ in the evolution equation for $T$, viscous stress $\sim v_{ij}$ in the evolution equation for $\rho v_{ij}$, and a term $\sim \nabla_k \pi_{ik}$, in the equation for the displacement, 

$$\frac{\partial}{\partial t} U_i - v_i = -\beta \nabla_k \pi_{ik}. \quad (13)$$

(Analogous to heat-conductivity, $\beta$ quantifies the strength of the dissipation. Taking it as a scalar is an approximation.) All these terms serve the sole purpose of restoring the respective equilibrium conditions: 

$$\nabla^2 T, v_{ij}, \nabla_k \pi_{ik} = 0.$$

The dissipative “displacement rate” $\sim \nabla_k \pi_{ik}$, as a necessary result of thermodynamics, has been first recognized in the classical 1972-paper: “The unified hydrodynamic theory for crystals, liquid crystals, and normal fluids”, by Martin, Paraodi and Pershan [98]. It drives the system, boundary conditions permitting, toward a constant stress. If the stress is not constant, such as in elastic waves, it contributes to wave damping. If one concentrates on the evolution of constant stresses, this term vanishes and is irrelevant. However, if the energy is concave, this term wacks havoc by driving the system away from uniform stresses. Writing it in the notation of the 6x6 matrix, Eq. (9), as: 

$$\nabla_k \pi_{jk} \rightarrow \nabla_k \pi_{\alpha} = \partial \pi_{\alpha} / \partial u_{\beta} \nabla_k u_{\beta}, \quad (14)$$

we see that, if the matrix $\partial \pi_{\alpha} / \partial u_{\beta}$ has a negative eigenvalue, the corresponding term will its flip sign. Instead of keeping the stress uniform, it drives the stress towards non-uniformity. This in turn accelerates mass points, possibly leading to non-uniform velocities $v_{ij}$ and thus finite strain rates, $v_{ij} \equiv -\dot{\epsilon}_{ij}$. Initially, the stress perturbation will grow along the direction associated with the negative eigenvalue, but for finite times, this is by no means true, as the system will try to move towards a stable equilibrium state, whatever that is. See the next two sections what happens in granular matter. 

Eq. (13), in term of the elastic strain, Eq. (3), reads 

$$\frac{\partial}{\partial t} u_{ij} - v_{ij} \equiv -\frac{\partial}{\partial \lambda} \varepsilon^P_{ij} + \dot{\epsilon}_{ij} \quad (15)$$

$$-\nabla_i [\beta \nabla_k \pi_{jk}] \equiv p_{ij}, \quad (16)$$

where the double-arrow indicates the (non-symmetric) counterpart of the preceding term. Eq. (15) seems to suggest that the dissipative term $p_{ij}$ is simply the plastic strain rate, $p_{ij} \equiv \varepsilon^P_{ij}$, which apparently exists even in solid if the stress is nonuniform. This would be a confusing nomenclature, as none of the typically plastic phenomena such as connected to concepts of plastic potentials or flow functions (see Refs. [77,78] are addressed here, in the context of elasticity. The term plastic strain rate is more appropriate for the dissipative contributions discussed in the next two sections, on transient elasticity and granular media.

Note that heat diffusion and viscous stress exist in any system, in which entropy and momentum are state variables: liquids, solids, granular media, irrespective of the microscopic interaction. Same holds for the dissipative term $p_{ij}$, which exists in any system in which the elastic strain is a variable. This is the reason it also exists in granular media. Generally speaking, every dissipative term strives to satisfy its equilibrium condition by changing the value or distribution of the associated state variable. Equilibrium is achieved if all equilibrium conditions are satisfied, as entropy is then maximal.

### 2.2 Transient elasticity and plasticity

There are many transiently elastic systems in nature. If quickly deformed, they are elastic and capable of restoring their original shape. But this does not happen if the deformation is kept longer; then the deformation is irreversible, plastic. One example are polymeric melts that consist of entangled elastic strands, which elastically deform, but disentangle if given enough time. This leads to a reduction, and eventually vanishing, of the elastic stress. For such systems, the equilibrium condition is: 

$$\pi_{ij} = 0, \quad \text{or, equivalently} \quad u_{ij} = 0. \quad (16)$$

Consequently, the evolution equation (15) takes the form: 

$$\varepsilon^P_{ij} = \frac{\partial}{\partial \lambda} u_{ij} - v_{ij} \equiv -\frac{\partial}{\partial \lambda} \varepsilon^P_{ij} + \dot{\epsilon}_{ij} = -\lambda_{\varepsilon} u_{ij}, \quad (17)$$

with the plastic strain rate now a relaxation term, with a positive coefficient $\lambda_{\varepsilon}$. Employing essentially this equation, including the convective terms of Eq. (1), a wide range of polymer behavior including shear thinning/thickening and the Weissenberg or rod-climbing effect were reproduced [101,102].

It is noteworthy that the plastic strain rate in the form $\varepsilon^P_{ij} = -\lambda_{\varepsilon} u_{ij}$ is a diagonal Onsager term, hence off-diagonal ones such as 

$$\varepsilon^P_{ij} = \frac{\partial}{\partial \lambda} u_{ij} - v_{ij} = -\lambda \varepsilon^P_{ij} - p_{ijkl} v_{kl} \quad (18)$$

are also permitted. They will turn out to be useful in granular physics.

The close link, even identity, between transient elasticity and strain relaxation on one hand, and plastic behavior of irreversible shape change on the other, is
a useful insight. Similarly useful is the understanding
of the difference between elasticity and transient elas-
ticity. For the latter to be in equilibrium, the elastic
stress has to vanish, while a constant stress suﬃces for
the former. For verbal clarity, we denote

\[ \text{elastic equilibrium : } \nabla_i \pi_{ij} = 0, \]

“plastic equilibrium” : \( u_{ij} \equiv -\varepsilon_{ij}^* = 0, \)

(19)

where “plastic equilibrium” is short for “transiently
elastic, long-term equilibrium”.

There is a further subtlety that we must address
here. If the polymer energy depends on both the den-
sity and the elastic strain, there are two contributions
in the stress: the pressure as given by Eq. (12) and the
elastic stress. Then the system may possess an equili-
brum pressure even when Eq. (19) holds. However, if the
density is not an independent state variable, implying
\( P \equiv 0 \), an equilibrium pressure needs a finite \( \Delta \equiv -u_{ii} \)
to be sustained, and \( u_{ij} = 0 \) cannot be the equilibrium
condition. Rather, it is given as

\[ u_{ij}^* \equiv -\varepsilon_{ij}^* = 0, \text{ implying } \varepsilon_{ij}^p = -\lambda_e u_{ij}^*, \]

(20)

the vanishing of the deviatoric part, while the trace \( \Delta \),
not independent from the density, simply follows the
dynamics of the density. It does not relax.

Note that the relaxation time of \( \Delta \) and \( u_s \) need not
be the same. If that of \( \Delta \) is especially long, it may be
neglected for certain phenomena, for which the dynam-
ics is governed by \( \varepsilon_{ij}^p = -\lambda_e u_{ij}^* \) alone.

When the system is crossing an inflection surface,
the term \( -\lambda_e u_{ij} \), in Eq. (17) is not aﬀected, and con-
tinues to push the elastic strain toward \( u_{ij} = 0 \).

2.3 Granular matter

GSH was set up in compliance with thermodynamics
and conservation laws. Here, we discuss its structural
part, necessary if one is to be consistent with the general
principles of physics. In Sec. 3, a reduced complete ver-

derion of GSH, including only some constitutive choices,
is presented, which will be employed later to study the
jamming and un-jamming dynamics.

Two basic pieces of physics characterize granular
media: (1) They have two entropies: \( s_g \) for the granular
degrees of freedom and \( s \) for the much more numer-
ous microscopic ones. (2) Depending on circumstances,
granular media may be elastic or transiently elastic.
Both elastic and plastic equilibria of Eqs. (19) are there-
fore relevant. However, note that the equilibrium (limit)
state is not necessarily ever reached, neither under per-
manent deformation, nor under free relaxation. In the
former case, the system is permanently pulled away
from the equilibrium (steady state is \textit{not equal to equi-
librium}), while in the latter, if \( T_g \) relaxes fast enough,
the equilibrium cannot be realized by the other state
variables either.

Including \( s_g \) as an extra state-variable, with \( T_g \equiv \partial \mathcal{W}/\partial s_g \), the equilibrium condition is \( T = T_g \), obtained
by maximizing \( \mathcal{F} + s_g \mathcal{G} \approx \int \mathcal{F} \, \mathrm{d}V \), where \( s_g \ll s \) may be ignored. The equilibrium condition implies that
all degrees of freedom, microscopic as well as granular
ones, will eventually equilibrate with one another. Fur-
thermore, since for particles of grain size, one typically
has \( T_g \gg T \) by many orders of magnitude, \( \sim 10^{10} \),
we may set the equilibrium granular temperature to zero,
\( T_g = T \approx 0 \).

In analogy to the relaxation terms discussed above, the evolu-
tion equation for \( s_g \) must therefore possess a relax-
tation term \( \sim T_g \), pushing \( s_g \) towards \( s_g \propto T_g = 0 \). This
dissipation/relaxation takes place due to colli-
sions, with rate \( \sim T_g \), or due to elasticity, with rate
\( \sim T_e \), or both. In addition, analogous to the viscous
heating term in the hydrodynamic theory of Newton-
ian fluids, which transfers kinetic energy into heat, via
\( \eta_v \varepsilon_{ij} \varepsilon_{ij} \equiv \eta_v \varepsilon^2 \rightarrow T \frac{\gamma}{\pi} \), there is a term that transfers
kinetic energy into “granular heat” , \( \eta_v \varepsilon^2 \rightarrow T_g \frac{\gamma}{\pi} s_g \).

Therefore, assuming \( \nabla_i T_g = 0 \), and ignoring gradients,
the evolution equation for granular energy reads

\[ T_g \frac{\partial}{\partial t} s_g = -\gamma T_g^2 + \eta_v \varepsilon^2 \]

(22)

with coefficient \( \gamma = \gamma(T_g) \) dependent on \( T_g \), and the
compressional viscosity neglected, like convective and
diffusive terms, for the sake of brevity. To be used in the
following, after some re-writing \( \text{the evolution equa-
tion for granular temperature reads:} \)

\[ b \rho \frac{\partial}{\partial t} T_g = -\gamma_1 T_g^2 + \eta_v \varepsilon^2 \]

(23)

The effective temperature \( T_g^* = T_g + T_e \) is discussed in
more detail below in Secs. 4.5 and 4.6.

For given deviatoric (shear) strain rate, \( \varepsilon_s = |\varepsilon_{ij}| \equiv |\varepsilon_{ij}^*| \), the steady state solution is given and discussed in
section 4.5 with the limit case for \( \gamma_0 \ll \gamma_1 T_g \), or
\( T_e \ll T_g \):

\[ T_g = T_g^{(ss)} = v_s \sqrt{\frac{2\eta_v}{\gamma}} = \eta_v \sqrt{\frac{\gamma_1}{\gamma_1}}. \]

\(^{10}\) Preempting the discussion in Sec. 3 to write down the
final evolution equation for \( T_g \), for reasons detailed in (69; 70)
[21], and partially in Sec. 3 we use:

\[ s_g = \rho b T_g, \quad \eta_v = \eta_1 T_g, \quad \gamma = \gamma_0 + \gamma_1 T_g, \quad \text{or, equivalently} \]

\[ \gamma = \gamma_1 (T_g + \gamma_0/\gamma_1) \equiv \gamma_1 (T_g + T_e) \equiv \gamma_1 T_g \]

in order to work with parameters that do not depend on \( T_g \)
anymore.

When inserting \( \rho b \) into Eq. (22) for energy, the time derivative
of this variable is neglected.
a result known to hold in granular gases\footnote{Note the difference in nomenclature: \( T_G \sim T_g^2 \propto v_s^2 \), see the text around Eq. \( \ref{eq:24} \).} up to moderate densities\cite{[10],[23]}. In this case, the system is in the rate-independent elasto-plastic regime, where the granular temperature is proportional to the strain rate. For diminishing \( T_g < T_s \) and \( \gamma_0 > \gamma_1 T_g \), we have an exponential and much faster decay, \( \frac{\partial}{\partial t} T_g \propto \gamma_0 T_g \), however, also here the steady state granular temperature persists and remains relevant, as \( T_g^{\infty} \approx (T_g^{\infty})^2 / T_s \), see section \ref{sec:2.5}.

Returning to the elastic strain \( u_{ij} \), we note that granular media are elastic for quiescent grains, \( T_g = 0 \), as slopes of sand-piles demonstrate. If the particles “jiggle”, \( T_g \neq 0 \), the elastic shear strain and stress will diminish, and eventually vanish: Tapping a vessel of grains (with a finite number) long (and strong) enough results in a flattened granular surface, like in transient elasticity. Combining both conditions of Eqs. \( \ref{eq:19} \), the evolution equation for the elastic strain contains both types of plastic strain rates, see also Eqs. \( \ref{eq:15} \), \( \ref{eq:18} \).\footnote{The third term depends in particular on the gradient of the elastic stress, see below and Refs. \[22\],[53],[54],[103],[85\].}

\[
\dot{\varepsilon}_{ij}^p = \frac{\partial}{\partial t} u_{ij} - \gamma_1 u_{ij} - p_{ij} \kappa \gamma_{ji} + p_{ij}, \tag{24}
\]

where the first term on the right, pushing \( u_{ij} \) towards the plastic minimum \( u_{ij} = 0 \), operates only for \( T_g \neq 0 \).

The second term represents strain- or stress-driven plastic deformations – occurring well within the macroscopic, elastically stable regime, involving possibly local events, on the particle scale – and will be split up into an isotropic (volumetric) and a deviatoric (shear) contribution, \( p_e \) and \( p_s \), with the respective plastic deformation probabilities, see subsection \ref{sec:2.1}. The mechanistic origins of these probabilities, are not addressed here, rather see Refs. \[22\],[53],[54],[103],[85\] and references therein, where it is shown that (finite) granular systems can remain elastic for tiny strain, then have localized plastic events at larger strain, with probability increasing, before (global) yield takes place with particular probabilities as cast into a meso-scale, stochastic master-equation approach, in Refs{[104],[105]}.

The third term depends in particular on the gradient of the elastic stress, see below and Refs. \[59],[70\].

This plastic strain rate, \( p_{ij} \), pushes \( u_{ij} \) towards the elastic equilibrium of uniform stress in the energetically convex region, and away from it in the concave one, since the gradient of stress changes sign at the transition.

2.3.1 Dynamics at constant strain or stress

Equation \( \ref{eq:24} \), in addition to the dynamics of \( T_g \), Eq. \( \ref{eq:22} \), render granular behavior rather more complex than the superposition of behavior from polymers and elastic media. Imposing either a constant shear rate or a constant elastic stress in a polymer melt, Eq. \( \ref{eq:17} \), the steady state result is the same, \( v_s = \lambda_s u_s \), in either case. This symmetry does not hold for granular media – not even for the simplest case with \( T_e = 0 \), and \( p = 0 \).

This symmetry does not hold in grains. A constant shear rate \( v_s \), with the stationary solution \( T_g = v_s \sqrt{\eta_0 / \gamma} \) (for \( T_e = 0 \)) inserted into Eq. \( \ref{eq:24} \), ignoring the \( p \)-terms on the r.h.s., leads to a rate-independent evolution equation for \( u_{ij} \) that possesses the hypo-plastic structure\cite{[106]}. It accounts well for elasto-plastic motion\cite{[107]}, including the approach to the critical state and shear jamming\cite{[108],[109],[70],[71]}. On the other hand, holding the stress/elastic strain constant, and inserting the stationary limit of Eq. \( \ref{eq:24} \), \( v_s = \lambda T_g u_s \), into Eq. \( \ref{eq:22} \), yields the relaxation rate:

\[
\gamma_e = (\gamma_1 + \eta_0 \lambda^2) u_s^2 \approx \gamma_1 u_s^2, \] 

negative if \( u_s < u_s^c = \sqrt{\gamma_1 / \eta_0 / \lambda} \), we find \( T_g \) to relax, pushing the system into a static state. The relaxation rate vanishes (i.e., the relaxation time diverges) as the stress (or elastic strain) approaches the critical value and, with a further increase, the rate flips sign to positive above the critical value, see \[70],[71\], creating an ever increasing strain rate \( v_s \). Accordingly, switching from an imposed shear rate (say during an approach to the critical state) to an imposed sub-critical stress will render the system static due to the relaxation of \( T_g \), whereas a critical or super-critical stress will create \( T_g \) and thus accelerate the flow, since \( v_s \propto T_g \).

2.3.2 Dynamics in the concave region

Within the cone of Fig. \[1\] in the energetically convex region, as long as one considers only the evolution of uniform stresses, the elastic dissipative term \( p_{ij} = \nabla_i [\beta \nabla_j \pi_{jk}] + (i \leftrightarrow j) \) remains zero. Serving to maintain stress uniformity, it may simply be neglected. Yet this term wreaks havoc if the energy is concave.

Perturbing the system by a (local) stress, \( \delta \sigma_{ij} \), from a static situation, in the convex, stable region, results in a relaxation of the elastic strain, due to the sign of \( p_{ij} \). In contrast, in the concave region, because of Eq. \( \ref{eq:14} \), this relaxation turns into an explosion, and drives the stress towards further, stronger non-uniformity.

This accelerates the grains, locally, leading to nonuniform velocities \( v_i \) and finite strain rates, \( v_{ij} = -\dot{\varepsilon}_{ij} \neq 0 \). The latter serve as a source for granular heat, see Eq. \( \ref{eq:22} \), and create considerable \( T_g \), which activates the first plastic term of Eq. \( \ref{eq:24} \), which relaxes the stress back into the stable, convex region. Hence, although the imposed perturbation creates a local stress response along the direction associated with the negative eigenvalue initially, it is the stress relaxation back
to the convex region that dominates for finite times. If not strong/ fast enough, the system will yield or un-jam dynamically. This is one way how GSH accounts for stability and un-jamming dynamics by instability, both mediated by the granular temperature.

Unfortunately, including the elastic dissipative terms renders Eq. (24) an unstable partial differential equation, the solution of which requires increased technical efforts. This is undesirable in a first, qualitative study, and an approximation scheme may prove useful.

We suggest to go on neglecting the elastic dissipative terms, and to add a stress term to Eq. (22), such that $T_g$ is directly produced by an elastic stress.

The balance equations for $s, s_g$, for the energetically convex region, are given as

$$T \frac{\partial}{\partial t} s = R = \gamma T_g + \beta_i \pi_{ij} \pi_{kl} + \cdots ,$$

(25)

$$T \frac{\partial}{\partial t} s_g = R_g = -\gamma T_g + \eta_i v_i v_j ,$$

(26)

The equally permissible alternative was not adopted,

$$T \frac{\partial}{\partial t} s = \gamma T^2 + \cdots ,$$

(27)

$$T \frac{\partial}{\partial t} s_g = -\gamma T_g + \eta_i v_i v_j + \beta_i \pi_{ij} \pi_{kl} ,$$

(28)

because any static $\pi_{ij}$ would then produce $T_g$, leading to its decay. This is not observed. Yet the reasoning is not valid outside the cone, where static stresses are not stable. Hence we combine Eq. (25) with (28), noting

$$\beta_{ijkl} = 0 \text{ inside, and } \beta_{ijkl} = 0 \text{ outside} ,$$

(29)

the cone. The explicit form for $\beta_{ijkl}$, $\beta_{ijkl}$ is a constitutive choice that will be given in the next section. In the notation of Eq (23), we have

$$b \frac{\partial}{\partial t} T_g = -\gamma_1 T_g^2 T_g + \eta_1 v^2_s + \beta_ijkl \pi_{ij} \pi_{kl} .$$

(30)

3 Granular solid hydrodynamics (GSH)

GSH is a continuum mechanical theory for granular media, set up in compliance with thermodynamics and conservation laws. GSH possesses the state variables: (i) density, $\rho$, or volume fraction, $\phi = \rho/\rho_p$, (ii) momentum density, $\rho v_i = 0$, neglected here, (iii) elastic isotropic strain $\Delta = -w_t = \epsilon'' = \ln (\rho/\rho_1)$, (iv) elastic deviatoric (shear) strain $u_{ij} = \sqrt{2} \Delta_{ij}$, (v) granular temperature $T_g \propto T^2_g$, and (vi) temperature $T$, not used in the following, with conventions and nomenclature given in Sec. [1.5].

The question is now if it is possible to catch the complex phenomenology at yielding, jamming, un-jamming, elasticity and loss of elasticity with a simple model that only knows about four state variables: $\rho, \Delta, u_s$, and $T_g$. For the sake of completeness, we first recollect the more complex, more complete classical GSH, as published in the previous years, in Sec. 3.1 before we reduce GSH to an over-simplified minimal model in Sec. 3.2 which will allow for a better understanding of the structure of GSH. Note that the nomenclature of classical GSH is applied in Sec. 3.1 whereas we switch to the positive compressive strain convention and nomenclature in Sec. 3.2.

3.1 About classical GSH

The complete equations of GSH may be found in Refs. [69,70], a simplified version in Ref. [71], from which we boil down to a minimalistic version in subsection 3.2 ignoring not only momentum density and gradients, but also the density dependence of most transport coefficients and parameters, since those represent constitutive assumptions, rather than basic theory. First, we discuss a few complications in the classical GSH nomenclature, that are not necessary for our present focus, but will become important if a more quantitative model is the goal, so that we keep them as reference for the sake of completeness.

3.1.1 The classical GSH constitutive model

The energy density has a thermal and an elastic part:

$$w = w_T + w_\Delta , \quad w_T = s_g^2 / (2 \rho b) ,$$

$$w_\Delta = \sqrt{\Delta (2 B (\rho) \Delta^2 / 5 + G (\rho) u_s^2)} , \quad B, G > 0 ,$$

(31)

with $P_\Delta = \pi_{ef} / 3$. This represents the first constitutive assumption at the core of classical GSH. In the following, we drop the $\rho$-dependence of $B$ and $G$ for convenience. (In previous GSH-publications, $G$ was denoted as $A_\gamma$.) The elastic stresses are defined as the derivatives of $w$ with respect to the elastic strain $u_{ij}$:

$$\pi_{ij} = -\partial w / \partial u_{ij} = P_{\Delta} \delta_{ij} - \pi_s u_s^* / u_s ,$$

(32)

$$P_\Delta = \sqrt{\Delta (B \Delta + Gu_s^2 / 2 \Delta)} , \quad \pi_s = 2G \sqrt{\Delta} u_s ,$$

(33)

$$4 P_\Delta / \pi_s = 2 (B/G) (\Delta / u_s) + u_s / \Delta ,$$

(34)

which represents no constitutive assumption, but is just a consequence of Eq. (31). Like the elastic stress, being conjugate to the elastic strain, the granular temperature is conjugate to the granular entropy, which allows to define the thermal pressure, $P_T$, as the derivative of the thermal free energy with respect to volume, at constant $T_g$, as:

$$T_g \equiv \partial w_T / \partial s_g = s_g / \rho b , \quad w_T = \rho b (T_g^2 / 2) ,$$

(35)

$$P_T \equiv - \left[ \frac{\partial w_T / T_g}{\partial [1/\rho]} \right]_{T_g} - \frac{\rho b T_g^2}{2} \frac{\partial b}{\partial \rho}$$

(36)
where we note that the granular entropy is not needed, replaced by the density dependent function \( b = b(\rho) \). The elastic energy \( w_{\Delta} \) has been tested for: (1) static stress distributions in silos, sand piles, point loads on a granular sheet \([10]\); (2) incremental stress-strain relations from varying static stresses \([11]\); (3) propagation of elastic waves at varying stresses \([12]\).

As already observed in Ref. \( [69] \), \( w_{\Delta} \) is convex if:

\[
\frac{u_s}{\Delta} \leq \sqrt{2B/G} =: g_c, \quad \text{or (37)}
\]

\[
\pi_s \leq \frac{P_\Delta}{2\sqrt{2B/G}} = 2/g_c .
\]

Because the macroscopic friction, or yield limit, \( \mu_0 := \sqrt{2B/G} \), is observed to be not (or only weakly) density dependent, at least for cohesionless granular media, the next constitutive model assumption used is: \( G/B = \text{const.} \), and

\[
B = B_0 \left[ (\rho - \bar{\rho})/(\rho_{cp} - \rho) \right]^{0.15}, \quad \text{(38)}
\]

where \( B_0 > 0 \) is a constant, and \( \bar{\rho} \equiv \frac{4}{3}(20\rho_{tp} - 11\rho_{cp}) \), with \( \rho_{cp} - \rho_{tp} \approx \rho_0 - \bar{\rho} \). \( \rho_{cp} \) is the random-close packing density, the highest one at which grains may remain uncompressed, \( \rho_{tp} \) is the random-loose packing density, the lowest one at which grains may stay static.)

The expression for \( B \) was empirically constructed to account for three granular characteristics: (1) It provides concavity, for any density smaller than \( \rho < \rho_{tp} \), and convexity between \( \rho_{tp} \) and \( \rho_{cp} \), ensuring the stability of elastic solutions in this region. (2) The density dependence of sound velocities, \( c \) (as measured by Hardin and Richart \([113]\), is well approximated by \( c = \sqrt{B/\rho} \approx \sqrt{B\Delta^{1/2}/\rho} \). (3) The slow divergence at \( \rho_{cp} \) mimicks the fact that the system is much stiffer for \( \rho = \rho_{cp} \) than at loose packing \( \bar{\rho}(\rho = \rho_{tp}) \). Comparing these constitutive assumptions for \( G \) and \( B \) with particle simulations is subject of ongoing work, but goes beyond the scope of this paper \([13]\).

Finally, the function \( b \) was chosen as:

\[
b = b_1 + b_0 \left[ 1 - (1 - \rho/\rho_{cp})^a \right], \quad \text{(39)}
\]

with another small power law, \( a \approx 0.1 \), such that \( P_T \approx w_T \rho \) for \( \rho \to 0 \), and \( P_T \approx w_T/(\rho_{cp} - \rho) \) for \( \rho \to \rho_{cp} \), limits which reduces \( b \) to first or second term, respectively,

\[P_T = \frac{\rho^2 T_0^2}{2} \left[ \frac{b_1}{\rho^2} + \frac{ab_0}{\rho_{cp}(1 - \rho/\rho_{cp})^{1-a}} \right] =: \rho g_p T_0^2 , \quad \text{(40)}\]

which defines the abbreviation \( g_p = (\rho/2)\partial b/\partial \rho \), that also is set to constant in the following sections, which is only a good approximation for low densities, i.e., \( g_p \approx b_1/2 \approx 1 \).

3.1.2 The evolution equations

For completeness, we specify the evolution equations in the classical GSH nomenclature, where we note the sign conventions \( \Delta = \varepsilon_\ell^\prime, \alpha_{ij} = -\varepsilon_i^\prime j \), and \( v_{ij} = -\dot{\varepsilon}_{ij} \), see Sec. 1.5.

For the elastic strain one has:

\[
\frac{\partial}{\partial t} u_{ij} = v_{ij} - \lambda T_g u_{ij}^*, \quad \text{(41)}
\]

\[
\frac{\partial}{\partial t} \Delta + v_{ij} = \alpha_{ij} u_{ij}^* - \lambda_1 T_g \Delta , \quad \text{(42)}
\]

with \( \alpha_{ij} \) as an off-diagonal Onsager coefficient, accounting for the Reynolds dilatancy. Mass and momentum conservation read:

\[
\frac{\partial}{\partial t} \rho + \nabla_i (\rho v_i) = 0 , \quad \text{(43)}
\]

\[
\frac{\partial}{\partial t} (\rho v_i) + \nabla_i (\sigma_{ij} + \rho v_j v_{ij}) = -\rho g_i , \quad \text{(44)}
\]

with the total stress: \( \sigma_{ij} = \pi_{ij} + P_T \delta_{ij} - \eta_l T_g u_{ij}^* \), with viscosity, \( \eta_l = \eta_l T_g \).

Finally, the evolution equation for \( T_g \), with \( b \) as given by Eq. \((35)\), and \( T_g^* \equiv T_g + \gamma_0/\gamma_1 =: T_g + \tau_e \), is given by Eqs. \( (39)\).

The coefficients \( \alpha_{ij}, \gamma_0, \gamma_1, \eta_l \), and \( pb \) are all functions of the state variables, especially the density, which would require many more constitutive assumptions, so that they are over-simplified and taken as constants in the following.

3.2 Minimal GSH type model for a material point

At the core of GSH, assuming a homogeneous representativ volume, without convection, \( \rho v_i = 0 \) and gradients, \( \nabla_i (\ldots) = 0 \), one has a postulated energy density,

\[
w = w_c + w_T , \quad \text{(45)}
\]

with an elastic and a dynamic, kinetic/granular contribution. The total stress is thus not an independent (state) variable, but can be abbreviated as

\[
\sigma_{ij} = \pi_{ij} + P_T \delta_{ij} + \sigma_{ij}^{\text{visc}} . \quad \text{(46)}
\]

\[=: P_\Delta \delta_{ij} + \pi_{ij}^* + \rho T_g^2 g_p \delta_{ij} + \chi \dot{\varepsilon}_i \dot{\varepsilon}_j + \eta \dot{\varepsilon}_{ij} , \]

12 To account for the un-jamming transition at the random loose density, \( \rho_{tp} \), a density dependence of \( B \) was seen as necessary in the classical GSH literature. To account for the virgin consolidation curve, higher order elastic strain terms in the energy were proposed, with density dependent coefficients, see \([69,113]\). The Coulomb yield could be accounted for with no density dependence, as in Eq. \((17)\). Since our illustrative examples are focused on the latter, hence \( B \) is set to constant in Secs. 3 and 4. A quantitative comparison with particle simulation data will show which assumptions or terms are really needed.
where the five terms represent isotropic and deviatoric elastic stresses, kinetic/granular stress (with an over-simplified \( q_0 = 1 \), which should depend – at least – on density, see Eq. (40)), and isotropic and deviatoric viscous stresses, with viscosities \( \chi = \eta_v \) and \( \eta = \eta_s \), respectively.

3.2.1 The elastic stress

One can derive the elastic stress \( \pi_{ij} = \partial w / \partial u_{ij} \), from the simplest (non-linear) elastic energy density:

\[
\pi_{ij} = \partial w / \partial u_{ij} = B \Delta^{3/2} + \frac{1}{2} G u_s^2 \Delta^{-1/2} := B \Delta \Delta, \\
\]

and the deviatoric elastic stress is:

\[
\pi_{ij}^e := \partial w / \partial \varepsilon_{ij}^e = 2 G \Delta^{1/2} \varepsilon_{ij}^e := G \Delta \varepsilon_{ij}^e, \\
\]

implicitly defining the (\( \Delta \)-dependent) bulk and shear secant moduli \( B_\Delta \) and \( G_\Delta \), which mimic a linear \( \Delta \)-or \( \varepsilon_{ij}^e \)-dependence of isotropic or deviatoric stress, respectively, not to be confused with the (true) tangent moduli \( B, G \) and \( A \). The notation details and alternative definitions of the state variables \( \varepsilon_{ij}^e = \Delta \) and \( \varepsilon_{ij}^{*e} = -u_{ij}^* \) are given in Sec. 3.3.

3.2.2 Simplest GSH equations and discussion

For a material point, in absence of gradients, using \( \partial f / \partial t \sim \partial f / \partial \varepsilon \), the evolution of density with strain rate:

\[
\partial_\varepsilon \rho = \rho \varepsilon_v, \\
\]

has no free parameters. Here, positive strain-rate corresponds to compression and negative to extension, i.e., density increase and decrease, respectively; density can also be seen as the volume fraction, related to each other by the (constant) material density, i.e., \( \phi = \rho / \rho_0 \). Later, units will be chosen, such that \( \rho_0 = 1 \).

In the evolution equation for the isotropic elastic strain:

\[
\partial_\varepsilon \Delta = \varepsilon_v - \lambda_1 T_g \Delta + \alpha_1 \varepsilon_{ij}^{*e} \varepsilon_{ij}^e, \\
\]

the first term couples elastic and total strain together, while the second term is relaxing \( \Delta \) towards zero \[^{13}\] in case of finite \( T_g \), with rate \( \lambda_1 T_g \). The third term can be positive (or negative, e.g., at strain reversal) and thus works against (or with) the relaxation term, with rate \( \alpha_1 v_s = \alpha_1 |\dot{\varepsilon}_{ij}^e| \).

The third equation defines the evolution of the deviatoric (shear) elastic strain

\[
\partial_t \varepsilon_{ij}^{*e} = \dot{\varepsilon}_{ij}^e - \lambda T_g \varepsilon_{ij}^{*e}, \\
\]

where the first term creates deviatoric elastic strain, co-linearly with the strain-rate, while the second term relaxes the deviatoric elastic strain, with rate \( \lambda T_g \). A dilatancy term analogous to the third in Eq. (49) is not required by the Onsager relation, but may be added for symmetry, as was done in Ref. \[^{81}\].

The fourth equation represents the evolution of the granular temperature

\[
\partial_t T_g = -R_T T_g T_g^* + f_T(\dot{\varepsilon}_{ij}^e), \\
\]

with the abbreviation for the dissipation rate \( R_T = \gamma_1/(\rho b) = R_{T0}(1 - r^2) \), proportional to the energy dissipation factor \( (1 - r^2) \), where \( r \) is the (effective) restitution coefficient. The energy creation terms are condensed into the tensor function \( f_T(\dot{\varepsilon}_{ij}^e) \), independent on \( r \), so that one could split them off with two energy creation rates, \( R_{T0} f_0^* = \eta_s / (\rho b) \) and \( R_{T0} f_0^e = \eta_v / (\rho b) \), for shear and volumetric strain-rates, respectively.

3.3 Minimal elastic model with two variables

One could decompose the elastic stress and strain tensors into invariants (and their orientations). Under the assumption of fixed and co-linear tensor-eigensystems, and ignoring the third invariant for the sake of brevity, what remains are the isotropic and deviatoric stresses, \( \sigma_\sigma = \{P_\Delta, \pi_s = \pi_s^* \} \), and elastic strains, \( u_\sigma = \{ \Delta, u_s = \}

\[^{13}\] Relaxation of \( \Delta \to 0 \), at fixed density, \( \rho \), implies that the granular temperature (jiggling) causes the jamming density to relax as \( D \to \rho \), in both jammed and un-jammed states, increasing and decreasing, respectively. A decrease (an increase) of the elastic strain, \( \Delta \), at fixed density, \( \rho \), corresponds to an increase (a decrease) of the jamming density, \( D \), see Ref. \[^{53}\]. On the other hand, at fixed confining pressure, \( P \), a jammed system, at finite, but small \( T_g \) (tapping) will develop to a state such that the elastic pressure, \( P_\Delta = P - P_T \approx P \), remains constant; relaxation of \( \Delta \) then corresponds to an increase of density, i.e., compaction.

\[^{14}\] After large strain, one has a positive product, \( \varepsilon_{ij}^{*e} \dot{\varepsilon}_{ij}^e > 0 \), but at strain reversal the same term can be negative, for a while, until the elastic deviatoric strain reverts direction.
\[ \delta^2 w = -\delta \pi_{ij} \delta u_{ij} = \delta \pi_a \delta u_a = \frac{\partial \pi_a}{\partial u_\beta} \delta u_\beta > 0. \] (52)

Using the (positive) invariants yields the simple 2x2 Gaussian matrix:

\[
\begin{pmatrix}
\delta \pi_n & \delta \pi_s \\
\delta \pi_s & \delta \pi_e
\end{pmatrix} = \begin{pmatrix}
B & A \\
A & G
\end{pmatrix} = C
\] (53)

If it has only positive eigenvalues, the (elastic) energy \( w_e \) is a convex function of the elastic strain-invariants \( \delta \) and \( u_\beta \). With other words, an elastic stability criterion is \( \det(C) = BG - A^2 > 0 \).

### 3.3.1 GSH with Hertzian type elasticity

In the special case of a Hertzian type energy density, see Eq. (47), as typically used in the GSH literature \[70\], one has:

\[ B = (3/2)B \Delta^1/2 - (1/4)G\nu^2 \Delta^{-3/2} \neq B_\Delta, \]
\[ G = 2G \Delta^1/2, \] and \( A = G \Delta^{-1/2} u_\beta. \)

With this, the stability condition, \( BG - A^2 > 0 \), translates to

\[ g_e^2 := 2B/G \geq (u_\beta/\Delta)^2, \] (54)

as previously shown in Eq. (12) in Ref. \[71\], and in Eq. (47) above, for elastic, static systems above jamming, for \( \Delta > 0 \), while \( w_e = 0 \) and thus \( \det(C) = 0 \) for \( \Delta \leq 0 \).

### 3.3.2 Eigen-values and -vectors at elastic instability

First, we compute the eigen-values and -vectors from the matrix \( C \), before we introduce constitutive assumptions and discuss those separately in the next subsections.

Basic linear algebra yields the two eigen-values, \( C_{1,0} = (B + G)/2 \pm \sqrt{(B - G)^2/4 + A^2} \), as solution of the quadratic equation \( 0 = (B - C)(G - C) - A^2 = C^2 - C(B + G) + BG - A^2 \), with \( C_1 = B + G \) and \( C_0 = 0 \), at the point of instability, where \( BG = A^2 \).

Using \( C_1 \), and \( A = \sqrt{BG} \), with the two equations

\[ -A_{11} + A_{22} = 0 \quad \text{and} \quad A_{11} - B_{11} = 0, \]
results in the corresponding eigen-vector (with \( n_2^{(1)} = n_1^{(1)} \, g/C = n_1^{(1)} \sqrt{G/B} \)), which defines the “direction” (in elastic strain invariants) of maximal stability: \( n_1^{(1)} = \pm(1, \sqrt{G/B})/\sqrt{1 + G/B} \).

Using \( C_0 = 0 \), and \( A = \sqrt{G/B} \), with the two equations

\[ B_{11} = 0 \quad \text{and} \quad A_{11} + B_{11} = 0, \]
results in the corresponding eigen-vector (with \( n_2^{(0)} = -\hat{n}_1^{(0)} \sqrt{B/G} = n_1^{(0)} \sqrt{B/G} \)), which gives the “direction” of instability (in the space of elastic strain invariants):

\[ \hat{n}_1^{(0)} = \pm(-\sqrt{B/G}, 1)/\sqrt{1 + G/B}, \]

and \( \hat{n}_2^{(0)} = 2 \pm (1, \sqrt{G/B})/\sqrt{1 + G/B} \).

Note the special role the ratio of shear to bulk modulus takes in this analysis.

### 3.3.3 Hertzian elastic energy instability

The non-zero eigenvalue can be re-written, using the choice for \( w_e \) in Eq. (47), as: \( C_1 = [B + 2G]\Delta^{1/2} = B[1/4g_e^2]\Delta^{1/2} \), with \( g_e = \sqrt{BG}/G \), while the zero eigenvalue will be more relevant for understanding the failure mechanism.

Using \( w_e \) in Eq. (47), this translates to the eigen-vectors: \( \hat{n}_1^{(1)} = \pm(g_e, 1)/\sqrt{1 + g_e^2} \), and \( \hat{n}_0 = \pm(1, -g_e)/\sqrt{1 + g_e^2} \). More explicitly, incremental changes in the elastic strain, \( \delta u_\alpha = (\delta \Delta, \delta u_\alpha) = \delta \epsilon \hat{n}_1^{(0)} \), at the point of elastic instability, can be done without any change of elastic energy, \( \delta^2 w = (\delta \epsilon^2)\hat{n}_1^{(0)} \hat{n}_2^{(0)} C_{\alpha \beta} = 0 \), and are thus permitted.

In a shear to normal stress space, one could see the limit of elasticity as one possible definition of the maximal failure mechanism. Using the ratio of shear to bulk modulus as an elastic strain increment will require energy to be realized. For energy considerations, see also Ref. \[55\] and references therein.

### 3.3.4 Anisotropic, linear elastic energy instability

In Ref. \[81\], the elements of the constitutive matrix \( C \) were directly deduced from particle simulations, and took a form (slightly simplified here by implying that the fabric and the elastic strain are proportional): \( B = B_0 \phi \), with the product of volume fraction \( \phi \) and coordination number \( \phi \Delta \), which is a non-linear function of \( \Delta \), \( G/B = G_\phi(\Delta)(1 - \nu_\phi) \), and \( A/B = \nu_\phi \).

From this, the condition for elastic instability translates to:

\[ w_e^2 = G_\phi(\Delta)(1 + G_\phi(\Delta)) \]

which implies a very narrow but steep elastic regime for small \( \Delta \), since \( G_\phi(\Delta) = (1/2)(1 - \exp(-\Delta/\Delta_0)) \rightarrow (1/2)\Delta/\Delta_0 \), vanishes for \( \Delta \rightarrow 0 \), so that \( w_e^2 \propto \sqrt{\Delta} \). For large \( \Delta/\Delta_0 \gg 1 \), one has instead \( w_e \approx 1/\Delta, \) independent of \( \Delta \).
The “direction” (in elastic strain invariants) of maximal stability becomes: \( \hat{n}^{(1)} = \pm (1, \sqrt{G/B})/\sqrt{1 + G/B} = \pm (1, u_s)/\sqrt{1 + u_s^2} \), and with the perpendicular “direction” of maximal in-stability: \( \hat{n}^{(0)} = \pm (-\sqrt{G/B}, 1)/\sqrt{1 + G/B} = \pm (-u_s, 1)/\sqrt{1 + u_s^2} \), after using \( \sqrt{G/B} = A/B = u_s \).

### 3.4 Special cases

In order to understand what the eigen-vectors mean, it is instructive to consider a few simple special cases. Some of these cases are later studied analytically and numerically. They represent simplifications that boil down a complicated theoretical framework to a simpler, possibly even transparent form that allows for better understanding and sometimes even for analytical solutions. We propose to apply those special cases to any new theory before one really applies the whole framework. Furthermore, the special cases allow to isolate a few of the terms and possibly calibrate the model parameters one by one.

For the rest of this section, we use the results from the Hertz-like elastic energy density, as discussed in subsection 3.3.3. Most of the cases are illustrated schematically in Fig. 2.

Fig. 2 Sketch of the (strain-rate driven) deformation cases in the space of the elastic strain invariants, i.e., \( u_s \) plotted against \( \Delta \). The numbers at the black arrows indicate the case-number, where dashed, thin lines are not allowed, continuing the trends in the permitted zone. The red arrows give the eigen-vector of instability \( \hat{n}^{(0)} \).

Except for the first case 0, the following cases start from a jammed, elastically stable state with finite initial elastic strains \( \Delta(0) > 0 \) and \( u_s(0) > 0 \).

- **(case 0)** Assume the system unjammed, \( \Delta(0) < 0 \), and apply a constant compressive strain-rate, \( \dot{\varepsilon}_v = -\dot{v}_{tt} > 0 \). The density and the elastic strain, \( \Delta = \log(\rho/\rho_J) \), will grow together until the system jams at \( \rho_J \), from which on its evolution equation kicks in. It was shown in Refs. [116][117], and earlier works cited therein, that already below jamming, the jamming density (and thus \( \Delta \)) depends on the procedure of preparation, in particular on the strain-rate and on the granular temperature, however, this fact goes beyond the present focus and is thus ignored here.

- **(case 1)** Assuming a purely isotropic de-compression, \( \dot{\varepsilon}_v = -\dot{v}_{tt} < 0 \), from a jammed state, one expects the elastic isotropic strain, \( \Delta \), to decrease faster than its deviatoric (shear) counterpart, \( u_s \), until at \( u_s^2 = (2B/G)\Delta^2 \), or \( u_s = g_e \Delta \), the system cannot sustain the applied shear-stress anymore, so that un-jamming due to instability with respect to shear occurs. In order to remain at least marginally stable, one needs a decrease of \( u_s \) to \( u_s^0 = g_e \Delta \), a situation that could be referred to as shear-yielding [59][56].

- **(case 1b)** In the situation without initial elastic shear strain, \( u_s(0) = 0 \), the stability criterion is always true and the system remains stable until isotropic un-jamming takes place at \( \Delta = 0 \).

- **(case 2)** In the case of isotropic compression, the model remains stable, unless the virgin consolidation line is reached, where the system restructures to be able to carry the increasing stress.

- **(case 3)** Assuming a purely deviatoric (volume conserving) shear strain rate, \( \dot{\varepsilon}_{ij}^* = -\dot{v}_{ij}^* \), from a state with initial \( \Delta > 0 \), one expects the elastic deviatoric (shear) strain, \( u_s \), to increase faster than its isotropic counterpart, \( \Delta \), could build up, until at \( \Delta = u_s/g_e \), the system cannot sustain pressure (isotropic stress) anymore, so that an instability with respect to volume change occurs, and one has a consequent increase of \( \Delta \to \Delta^0 = u_s/g_e \), which can be seen as one origin of dilatancy. However, the evolution of \( \Delta \) is changing qualitatively, when the limit of elastic stability is reached, as will be studied numerically later on.

- **(case 3b)** Under the same purely deviatoric deformation, the isotropic elastic strain \( \Delta \) could also decrease, which only leads to instability at smaller elastic strains, not much changing the considerations in case 3, but rather leading to compactancy.

Several of the cases discussed above will be now studied analytically (as far as possible) and numerically.

### 4 Analytical results for special cases

This section considers first the athermal limit, \( T_g = 0 \), before granular temperature is included into the equations and various versions of the model are discussed. Finally, two regularization schemes are proposed, to be
later used for the numerical solutions. But first we summarize the equations that will be used in this section.

The set of model equations is summarised here for reference, with the colored terms representing extensions from the black terms (representing model 0):

\begin{align}
\phi \frac{\partial \rho}{\partial t} &= \rho \frac{\partial \rho}{\partial t} \\
\partial_t \Delta &= \Delta(1 - p_v) - \lambda_1 T_g \Delta p_g + \alpha_1 \varepsilon_{ij}^{\ast} \varepsilon_{ij}^{\ast}(1 - p_s) \\
\partial_t \varepsilon_{ij}^{\ast} &= \varepsilon_{ij}^{\ast}(1 - p_v) - \lambda T_g \varepsilon_{ij}^{\ast} + \alpha_d \\
\partial_t T_g &= - R_g T_g^2 (T_g^* / T_g) + f_T (\varepsilon_{ij}) + f_g(g^*) ,
\end{align}

before some meaningful special cases (isotropic and deviatoric loading) are discussed below, for which analytical solutions are provided, if possible. The colored terms are not present in the original Eqs. [15 - 51], which is referred to as model 0, having thus no valid athermal limit.

The blue terms are hold-ers for elements discussed below, in subsection 4.6 or to be added in future, introduced in Refs. [118, 51]. The rate of cooling is modified in the elastic, jammed state (Δ > 0) by adding an "elastic dissipation rate" T_e, referred to as model e, as T_g^*/T_g = 1 + T_e/T_g = 1 + T_e/Δ^b/T_g, where only the special case h = 0, i.e., T_e = T_e0, will be treated below[15]. The presence of T_e does not affect the dynamics too much for large T_g (for more details, see below), but in the limit of very small T_g → 0, for elastic, jammed systems, this (phonon/wave-driven) dissipation becomes important, providing an exponential decay of T_g → 0 in absence of other driving mechanisms (and constant Δ > 0).

The new magenta term f_g(g^*) = f_g(g^*)^2θ(g^*), in Eq. [58], is only active if the system is outside of the elastically stable regime, where g^* = u_1 / Δ - g_e > 0, with the limit of elastic stability g_e, and the step-function θ(x ≥ 0) = 1, or θ(x < 0) = 0. This term generates more granular temperature, jiggling, due to concavity of the elastic energy, the more the system gets elastically unstable.

The terms (1 - p_v) and (1 - p_s) represent the probabilities for elastic deformations, with p_v and p_s the probabilities for isotropic/deviatoric plastic deformations, respectively, see Ref. [53], as specified in Sec. 2.1 and discussed next, in section 4.1.

4.1 The granular athermal limit T_g = 0

Enforcing the athermal case, T_g = 0, the system of equations reduces to:

\begin{align}
\partial_t \Delta &= \varepsilon_{ij} (1 - p_v) + \alpha_1 \varepsilon_{ij}^{\ast} \varepsilon_{ij}^{\ast}(1 - p_s) \ , \\
\partial_t \varepsilon_{ij}^{\ast} &= \varepsilon_{ij}^{\ast}(1 - p_v) ,
\end{align}

where the off-diagonal Onsager coefficients p_v and p_s were introduced in Ref. [60] and taken equal to \alpha_1. Alternatively, they were interpreted in Refs. [53, 54] as the probabilities for (isotropic and deviatoric) plastic (re-structuring) events in the packing. Note that in Eqs. [59] and [60], the probabilities for isotropic and deviatoric plastic deformations are attached to isotropic and deviatoric strain-rates, respectively.

4.1.1 Athermal isotropic loading

For isotropic loading (\varepsilon_{ij}^* = 0), the system reduces even further to \varepsilon_{ij}^* := \varepsilon_{ij} - \partial_t \Delta = \varepsilon_{ij} p_v, or \varepsilon_{ij}^* := \partial_t \Delta, with Δ = 0, the system of equations reduces to:

\begin{align}
\partial_t \varepsilon_{ij}^* &= \varepsilon_{ij}^* (1 - p_v) \ , \\
\partial_t \varepsilon_{ij}^{\ast} &= \varepsilon_{ij}^{\ast}(1 - p_v) ,
\end{align}

For frictionless spheres – and even more for realistic frictional non-spherical particles – we have to come up with a better relation for the probability for isotropic plastic rearrangements.

The probability for plastic deformations was reported in Ref. [53], as p_{\lambda}^\Delta \propto \Delta/\lambda_0, with the limit elastic strain, \lambda_0 = \ln (\rho/\rho_\infty), expected to be reached after infinitely many isotropic loading/un-loading cycles up to density \rho, with the corresponding density:

\begin{align}
\rho_\infty &= \rho_0 + b_\infty \left[ \frac{\rho}{\rho_0} - 1 \right]^{b_\infty} ,
\end{align}

with the half-sided linear function [x > 0]_+ = x, and [x ≤ 0]_+ = 0, otherwise, guaranteeing \rho_\infty = \rho_0 for

\begin{align}
\dot{\rho}_\infty &= \dot{\rho}_0 + b_\infty \left[ \frac{\rho}{\rho_0} - 1 \right]^{b_\infty} + ,
\end{align}

15 For a Hertzian type bulk modulus, the time-scale of momentum (wave) propagation, for u_0 = 0, can be estimated as

\begin{align}
t_e = 1/T_e = d/v_e \propto d / \sqrt{\rho B} \propto \sqrt{\Delta/\lambda} \propto \Delta^{-1/2} ,
\end{align}

i.e., an exponent h = 1/4. This estimate, together with a Hertzian elastic pressure, P_\Delta \propto \Delta^{3/2}, yields an estimated wave speed v_e \propto P_\Delta^{1/6} or moduli B \propto P_\Delta^{1/3}.

16 By using the chain rule, one has: \partial_t \Delta = (\partial_t \rho)/\rho - (\partial_t \rho)/\rho = \varepsilon_{ij} - \partial_t \ln (\rho/\rho_\infty) , as input.

17 Inserting the expression from above, this yields the athermal evolution of the elastic strain: \Delta = v_{\lambda}^{\Delta} - \lambda_1 T_{e0} \Delta^{1/2}. 

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The density $\rho_\infty$ is the density for which the system would isotropically jam/un-jam, where the subscript indicates infinitely long relaxation.\footnote{Thus, $\rho_\infty$ takes the place of the random close packing density, $\rho_{cp}$, but continuously grows with density. The high densities could be achieved (by over-compression) of soft particles (rubber, gel, etc.), whereas hard particles (metal, glass, etc.) would break (not considered here). For hard particles, one could replace Eq. (61) with a step function equal to $\rho_{cp}$ for $\Delta > 0$.}

For the sake of simplicity, in the numerical solution of the evolution equations, we implemented the simpler plastic deformation rate: $\dot{\varepsilon}_v p_0 = \rho_{cp} \max(\dot{\varepsilon}_v, 0)(\rho_\infty - \rho_f)/(\rho_\infty - \rho_{cp})$, with $\rho_{cp} = 1$, according to Kumar and Luding, 2016\footnote{This implies a relation $G_{\text{A}}/G_{\text{B}} = \mu_0^2 \Delta^x/u^c = 2G/[B + (1/2)(G(u^c_0/\Delta^c)^2)] = 4/[g_0^2 + (u^c_0/\Delta^c)^2]$ between shear and bulk modulus, and allows to determine from the quadratic equation: $\mu_0^2(u^c_0/\Delta^c)^2 - 4u^c_0/\Delta^c + \mu_0^2 g_0^2 = 0$ the shear to isotropic elastic strain ratio $u^c_0/\Delta^c = 2/\mu_0^2 \pm \sqrt{(1/2)\mu_0^4 g_0^2}$, with real solutions for $\mu_0^2 \leq 2/g_0^2$, as realized in cases modeled here (data not shown).}, idealized to be active for compression only, which yields qualitatively similar results, with a rather rapid approach to the maximal jamming density $\rho_\infty$, while the above model $\dot{\varepsilon}_v \rho_\infty^2$, with a much slower approach (stretched exponential) to $\rho_\infty$, will be detailed elsewhere. The evolution of the jamming density and of pressure with density during initial loading and cyclic un-/re-loading are plotted in Figs. 3 and 4, to illustrate the phenomenology, including un-jamming/jamming, with details of the (numerical) model given in the next Sec. 5.

\begin{equation}
\Delta > \rho \Delta \rho_{\text{max}} = 0.62 \text{ is un-jamming and jamming during the cycles for several times. The upper curve represents the elastic limit case, with } p_0 = 0, \text{ i.e., with no plastic rearrangements and the analytical pressure state-line: } P_\Delta = B\Delta^{3/2}, \text{ with } B = 1, \text{ for details see Sec. 5. The lowermost curves represent cyclic un-/re-loading from } \rho_{\text{max}} = 0.68 \text{ with amplitude } \delta p = 0.08, \text{ well below the jamming-point. The inset represents the void fraction } \varepsilon \text{ plotted against (logarithmic) } P, \text{ similar to Fig. 3a.}
\end{equation}

\subsection{Athermal deviatoric loading}

For purely deviatoric (isochoric) shear, $\dot{\varepsilon}_v = 0$, the elastic shear strain develops as $\partial_t \varepsilon_{ij}^e = \varepsilon_{ij}^{p^*} - \dot{\varepsilon}_{ij}^p$, equivalently, for the plastic strain rate $\dot{\varepsilon}_{ij}^p = \dot{\varepsilon}_{ij}^* - \partial_t \varepsilon_{ij}^e = \varepsilon_{ij}^{p^*} p_s$. Postulating the existence of a constant “critical” steady state for the stress ratio $\mu_0^2 := \mu_0^2(v_s \to 0) = \tilde{G}^* u^c_s/\Delta^c$\footnote{If one can assume: $\Delta \approx \Delta^c$, i.e., that the isotropic elastic strain is almost constant, close to its critical state limit already, Eq. (60) can be solved analytically, yielding an-
} this allows to express the probability for plastic (shear) events as:

$$p_s = \frac{\mu}{\mu_0^2} \approx \frac{\varepsilon_{ij}^{p^*} \varepsilon_{ij}^*}{u^c_s v_s} \approx \frac{u_s}{u^c_s},$$

where the last approximation is only valid after sufficiently long steady shear, close to the critical state, but not for strain reversal. The term in brackets is limited to keep it a probability, i.e., $[x > 0]_+ = x$, and $[x \leq 0]_+ = 0$, and thus also valid for strain-reversal, as done similarly in Ref.\cite{53,118,81} and references therein – based on, and in quantitative agreement with, DEM simulations.\footnote{The probability for plastic events in Eq.}
is finite, but very small, at the beginning of shear with build-up of elastic shear strain, $u_s$, but asymptotically approaches $p_s = 1$ for large strain in the perfectly plastic, critical state. At reversal of shear, the argument of the bracket-function becomes negative, i.e., the system is elastic with $p_s = 0$, until the shear strain is built up sufficiently in the new direction.\(^\text{21}\)

Fig. 5 Shear stress, $\sigma_{*\nu\nu} := \sigma_{*\nu}$, plotted against pressure, $P$, from simulations compressed up to $p_{\text{max}} = 0.61, 0.63, 0.65, 0.67$, and 0.69 (green dots), and subsequent cyclic pure shear with amplitude, $\delta \gamma \sim 0.28$, where the magenta dots represent the end-situation after six forward-backward shear cycles. The dashed line indicates the pre-set slope $\mu_0 = \sigma_{*\text{pre}}/P^\ast = 0.5$. The only other parameter active in this model is $\alpha_1 = 2$, where the case $p_{\text{max}} = 0.65$ was simulated with two other values of $\alpha_1 = 0.5$ and 8, to display the enhancing effect on pressure-dilatancy of this parameter. Note that the imposed maximal macroscopic friction, here, is chosen smaller than the elastic stability limit, $\mu_0 < 2/\mu_s = 1$, such that the latter is never reached.

Noting the similarity between $p_s$ and the $\alpha_1$-term, one can rewrite the evolution equation for the isotropic elastic strain as: $\Delta = \Delta^\ast + \alpha_1 u_s v_s (1 - p_s) \approx \alpha_1 u_s v_s (1 - p_s) = \alpha_1 u_s v_s$, for constant $v_s$ (not valid for strain-reversal). This equation has a critical state solution, $\Delta^\ast$, due to the term $(1 - p_s)$, as well as a stable elastic solution with $\Delta = 0$ for $p_s \approx 0$, see the infinite slopes in Fig. 5 for small shear strain and thus small shear stress. The deviatoric elastic strain evolves as $\dot{u}_s = v_s (1 - p_s) \approx v_s (1 - u_s/u_s^c)$, with analytical solution:

$$u_s(t) = u_s^c - [u_s^c - u_s(0)] \exp(-v_th/u_s^c)$$ \quad \quad (63)

with $u_s^c = \Delta [2/\mu_0^2 - \sqrt{(2/\mu_0)^2 - g_e^2}]$, as plotted in the inset of Fig. 5 as shear stress evolution $\pi_s^c = 2G\Delta^{1/2}u_s$. The exponential approach of $u_s$ to its critical state limit, see Ref. \(^\text{63}\), follows the discussion of which goes far beyond this paper.\(^\text{21}\)

This analytical solutions are very similar in form to those used in Refs. \(^\text{53,118}\), however, further discussion is beyond the scope of this paper.\(^\text{22}\)

4.2 The granular thermal limit $\dot{T}_y = 0$

Assume that one could maintain a constant granular temperature, which would result in the set of equations:

$$\partial_t \Delta = \dot{\varepsilon}_s (1 - p_s) - \lambda_1 T_y \Delta p_s + \alpha_1 \varepsilon_{ij}^* \dot{\varepsilon}_{ij}^* (1 - p_s) \quad (64)$$

$$\partial_t \varepsilon_{ij}^* = \dot{\varepsilon}_{ij}^* (1 - p_s) - \lambda T_y \varepsilon_{ij}^* \quad (65)$$

For vanishing strain-rate $\dot{\varepsilon}_{ij} = 0$, the equations decouple and only the relaxation terms survive. This corresponds to the “plastic equilibrium” limit case $\Delta = 0$, $\dot{\varepsilon}_{ij}^* = 0$, which is approached exponentially fast, with rates $\lambda_1 T_y$ and $\lambda T_y$. The term $p_s = 1$ allows to choose the plastic equilibrium of transiently elastic systems, for which $\Delta \to 0$, or in a form $p_s = 1 - \Delta/\Delta$, the granular plastic limit with $\Delta > 0$, see subsection 4.6.

For finite $\dot{\varepsilon}_{ij}$, the system will establish thermal, elasto-plastic dynamic states that are not discussed further for the sake of brevity.

Strictly controlling density, i.e., fixing $c$, the situation is interesting again for granular matter. Any perturbation, as tapping or small-amplitude cyclic shear, will typically result in a decrease of both the elastic strain, $\Delta$, and consequently the pressure, $P_\Delta = B_\Delta \Delta$, for which $u_s^c$ depends (weakly) on $\Delta$, the system of equations is still coupled and the analytical solution is only approximate.
4.3 Isotropic jamming in a minimal GSH

The model equations for isotropic compression/tension, with strain rates $\partial_t \rho = \dot{\varepsilon}_v \neq 0$, and $\dot{\varepsilon}_i^* = 0$, reduce to:

\begin{equation}
\partial_t \Delta = \dot{\varepsilon}_v (1 - p_v) - \lambda_1 T_g \Delta \tag{66}
\end{equation}

\begin{equation}
\partial_t \dot{\varepsilon}_i^* = -\lambda T_g \dot{\varepsilon}_i^* \tag{67}
\end{equation}

\begin{equation}
\partial_t T_g = -R T_g^2 (T_g/T_0) + f_T (\dot{\varepsilon}_i) + f_T (g^*) \tag{68}
\end{equation}

The density is coupled to strain-rate directly, while the second equation (67) is decoupled (just relaxing an existing elastic shear strain to zero). From the coupled evolution equations (66) and (68), for $\Delta$ and $T_g$, we observe that the situation at the end of an isotropic compression is independent of the density reached if $p_v = 0$. Of the evolution of $\Delta$ and could be (quantitatively) calibrated to the numerical data in Ref. [53] in a future study. The energy production term due to elastic instability in Eq. (68) would become active for finite $u_s$, when $\Delta < g_s u_s$, but is ignored here, assuming $u_s = 0$ (which is not strictly true in real systems, where there can be some small, random elastic deviatoric strain).

The evolution equation for $T_g$, abbreviating $\gamma = R_T = R_{T0} (1 - r^2)$, and assuming $T_c = 0$, results in an algebraic evolution:

\begin{equation}
T_g \frac{T_g}{T_g^0} = \frac{1}{1 + R_T T_g^0} \tag{69}
\end{equation}

in the free, homogeneous cooling state, as relevant for systems below jamming in the granular gas state. On the other hand, assuming the simplest model for $T_g^* \approx T_c$, with $h = 0$ (or for constant $\Delta$), for a small perturbation from an elastic base state, one has

\begin{equation}
T_g \frac{T_g}{T_g^0} = \exp (-RT_T \Delta t) \tag{70}
\end{equation}

as relevant for elastically stable systems, well above the jamming density, for which small perturbations decay exponentially fast.

For finite positive (compressive) strain-rate, the inhomogeneous solution leads to a divergent increase of $T_g$ with time due to the continuous energy input. For negative (expansive) strain-rate, the same is true, however, as soon as the system isotropically un-jams, the behavior should qualitatively change – which is not accounted for in the present version with constant parameters, in particular $f_s$ and $R_{T0}$; more details are beyond the scope of this study.

4.4 Pure shear from an isotropic state

This case was studied in detail by particle simulations in Refs. [51, 53], and should be studied analytically too with respect to questions about the build-up of anisotropy, and the degradation of moduli, but is skipped for the sake of brevity.

4.5 Steady state pure shear (model 0 and $\epsilon$)

In case of deviatoric pure shear, the density equation vanishes, since $v_\| = 0$ the density is conserved, $\partial_t \rho = 0$, and the terms with isotropic strain rate in the equations drop out. The remaining equations yield the steady state solution for the granular temperature:

\begin{equation}
\partial_t T_g = 0 = R_{T0} [-(1 - r^2) T_g T_0 + f_s^* \dot{\varepsilon}_i^* \dot{\varepsilon}_j^*] \tag{71}
\end{equation}

with $T_g^* = T_c + T_g$, so that (for $T_c = 0$):

\begin{equation}
(T_g^{(ss)})^2 + (T_g^{(ss)}) T_c - (T_g^{(ss)})^2 = 0 \tag{72}
\end{equation}

yields

\begin{equation}
T_g^{(ss)} = \pm \sqrt{(T_c/2)^2 + (T_g^{(ss)})^2} - T_c/2 \tag{73}
\end{equation}

where only the positive solution is reasonable. In the “collisional” limit $T_g \gg T_c$, one has the dynamic steady state: $T_g^{(ss)} \approx T_g^{(ss,0)} \propto \nu_s$, while for $T_g \ll T_c$, the steady state temperature in the “elastic” steady state is:

\begin{equation}
\nu_s^{(ss)} \approx (T_g^{(ss,0)})^2/T_c \propto \nu_s^2, \text{ i.e., it vanishes quadratically for } \nu_s \to 0. \tag{74}
\end{equation}

For the deviatoric elastic strain one has:

\begin{equation}
\partial_t \dot{\varepsilon}_i^* = 0 = \dot{\varepsilon}_i^* - \lambda T_g \dot{\varepsilon}_i^* \tag{75}
\end{equation}

so that:

\begin{equation}
\nu_s^{(ss)} = \nu_s / (\lambda T_g) \text{ and } \nu_s^{(ss)} = \sqrt{1 - r^2} / (\lambda f_s), \tag{76}
\end{equation}

while for the isotropic elastic strain one has:

\begin{equation}
\partial_t \Delta = 0 = -\lambda_1 T_g \Delta + \alpha_1 \dot{\varepsilon}_i^* \dot{\varepsilon}_j^* \tag{77}
\end{equation}

with elastic bulk-modulus $B_\Delta = B \Delta^{3/2}$. In this situation, the pressure curve shifts to smaller densities (larger $\epsilon$), and changes slope, both moving it away further from the elastic state-line (not shown here). On the other hand, large strain shear results in (pressure) dilatancy, shifting the state-line to the right, towards the VCL (but not beyond), defining the critical state line (CS) – see Fig. 6.
so that inserting Eqs. (71) and (73) yields the isotropic elastic strain in steady state:

\[ \Delta_{ss}(\sigma) = \frac{\alpha_1v_s^2}{\lambda_1(T_{ss}(\sigma))^2} \quad \text{and} \quad \Delta_0(\sigma) = \frac{\alpha_1(1 - r^2)}{\lambda_1\lambda_2} , \quad (74) \]

the former valid for model \( e \), the latter for the simplest model 0, where the subscript 0 indicates \( T_e = 0 \); model \( e \) is not indicated since it represents the default case.

In the “elastic” limit \( T_0 \ll T_e \), for \( v_s \to 0 \), the other two state variables, in model 0, behave as: \( u_s(\sigma) \to v_s^{-1}, \Delta(\sigma) \to v_s^{-2} \), and thus \( g_{ss}(\sigma) = u_{ss}/\Delta \to v_s \), i.e., a leading order linear increase with (shear) strain rate.

4.6 Steady state pure shear (model 1)

In model 1, only the evolution equation of the isotropic elastic strain has to be modified:

\[ \frac{d}{dt} \Delta = 0 = -\lambda_1T_0\Delta + \alpha_1u_{ij}^*e_{ij}^* \]

so that inserting Eqs. (71) and (73) yields the isotropic elastic strain in steady state:

\[ \Delta_{ss}(\sigma) = \frac{\alpha_1v_s^2}{\lambda_1(T_{ss}(\sigma))^2}g_{ss} = \frac{\Delta_{ss}(\sigma)}{P_0} , \quad (75) \]

for model 1 for constant or \( \Delta \)-independent \( P_0 \).

In some of the numerical implementations, we used \( p_0 = \Delta - \Delta_\infty \), in order to make \( \Delta \) relax towards a finite value, with \( \Delta_\infty = \log(\rho_\infty/\rho) \), as defined in Eq. (61).

This allows to re-write \( p_0 = \log(\rho_j/\rho_\infty) \), which makes the relaxation term vanish for \( \rho_j = \rho_\infty \), negative for larger values and increasingly positive for smaller jamming densities. Unfortunately, it also requires to solve a quadratic equation, resulting in

\[ \Delta_{ss}(\sigma) = (1/2)\Delta_\infty[1 + \sqrt{1 + 4(\Delta_{ss}(\sigma)/\Delta_\infty)}] , \]

i.e., an increased steady state elastic strain, representing strain-dilatancy. Note that this approach to achieve finite \( \Delta \) under steady state shear, increasing with density – as to be expected – is different in philosophy than making the bulk modulus factor \( B \) density dependent.

4.7 Discussion of the steady state

Dividing Eq. (73) by (74) yields the deviatoric to elastic strain ratio in steady state (in order to evaluate whether the system is elastically stable or not):

\[ g(\sigma) = \frac{\sqrt{2}}{\pi} \left[ \frac{\Delta_{ss}(\sigma)}{P_\Delta + P_T} \right] = \frac{\Delta_{ss}(\sigma)}{\Delta_{ss}(\sigma) + \Delta_{ss}(\sigma)} \quad (76) \]

If the ratio of elastic strains in Eq. (76) is smaller than the elastic stability limit \( g(\sigma) \leq g_e = \sqrt{2B/G} \) the system remains in a possibly stable (elastic, jammed) state, while it looses stability if the ratio reaches and/or exceeds the limit value.

Solving numerically the system of equations, including the transient evolution, confirms that the steady state is independent of the density, for model 0, see Sec. 5 as \( \rho \) does not appear in the steady state solutions above.

The elastic strain ratio, Eq. (76), which determines whether the system becomes elastically instable in steady state, is not the same as the macroscopic friction at which the material flows plasticly. Dividing the steady state shear stress by pressure defines the macroscopic (bulk) “friction”: \( \mu = \sigma^{*}/P \), which results in the steady state bulk friction:

\[ \mu(\sigma) = \frac{\sigma^{*}}{P} = \frac{\pi_{ij}^{*} + \eta v_{ij}^{*}}{P_\Delta + P_T} = \frac{\Delta_{ss}(\sigma)}{B} \frac{g_{ij}^{*}}{\Delta_{ss}(\sigma) + \Delta_{ss}(\sigma)} \quad (77) \]

In the slow strain-rate limit, \( \dot{\varepsilon}_{ij} \to 0 \), of Eq. (77), above jamming, \( \Delta > 0 \), the second terms in nominator and denominator vanish, linearly and quadratically with \( T_0 \to 0 \), respectively, and one has

\[ \mu(\sigma) = \frac{\Delta_{ss}(\sigma)}{B_\Delta \Delta_{ss}(\sigma)} \]

\[ \mu(\sigma) = \frac{2(G/B)(\Delta_{ss}(\sigma))^{-1}u_s^{*} + (1/2)(G/B)(u_s^*)^2{\Delta_{ss}(\sigma)}^{-2}}{4} \]

\[ \mu(\sigma) = \frac{4(G/B)(g(\sigma))^{2}}{g(\sigma) + 2} \quad (78) \]

For the special case \( g(\sigma) = g_e \), when the elastic limit of stability and the steady state ratio of elastic strains coincide, this translates to: \( \mu(\sigma) = 2/g_e \).

4.8 Temperature regularization (model g)

In order to regularize the elastic instability, we introduce a measure for the distance from the elastic limit

\[ g_s = (g - g_e) = (u_s/\Delta - \sqrt{2B/G}) \]

which can be used to regularize the temperature evolution

\[ \frac{d}{dt}T_s = R_T \left[ -T_s^2 + f_T(\dot{\varepsilon}_{ij}) + f_T(\theta_s)g_s \right] \quad (79) \]
with the step-function $\theta(g_e > 0) = 1$, and 0 else, so that one has for steady-state pure shear (with model 0):

$$
(T_g^{(ss)})^2 = \frac{f_s^2 U_g^2 + f_g \theta(g_e) g_e}{(1 - r^2)},
$$

(80)
i.e., just an elevated granular temperature that affects, in turn, the other state-variables (elastic strains) via their respective relaxation terms, as will be shown in the next section.[5]

5 Numerical solutions

In order to better understand GSH, we solve the system of equations numerically (with matlab, using ode45) and discuss the features of the simplest GSH type model without any constitutive assumption other than the form of the energy density in Eq. (47), but rather keeping all parameters constant, see table 1.

Units are chosen as $\rho_u = \rho_p = m_p/V_p = 2000$ kg m$^{-3}$, with mass, $m_p$, and volume, $V_p$, of a single particle, so that the dimensionless density is:

$$
\rho = (\rho_p/\rho_u) \phi = \phi, \quad \text{while time is measured in units of micro-seconds, } t_u = 1 \mu\text{s}, \quad \text{and length in units of particle diameters } d_u = d_p = 10^{-4}$$.m. With these choices, the unit of mass is $m_u = m_p = \rho_p V_p = (\pi/6)\rho_p d_p^3 = (\pi/3)10^{-9}$ kg, while stress and moduli have units of $\sigma_u = m_u/d_u/t_u^2 = (\pi/3)10^7$ kg m$^{-1}$ s$^{-2} \approx 10$ MPa.

The boundary conditions of the numerical solutions are first a preparation by isotropic compression, followed by pure deviatoric (volume-conserving) shear for large strain to approach the critical state, and finally a relaxation without any strain-rate.

5.1 Effect of density and dynamics

Next goal is to understand the behavior of the model at different densities and the effects of the elastic dissipation parameter $T_e$ and the temperature regularization $f_g$.

The initial preparation starts from an un-jammed state at $\rho(0) = 0.58$, and is applied up to different target densities $\rho = 0.61, 0.62, 0.63, 0.68, 0.74,$ and 0.80 during $t_p = 1000$. From this point on, pure shear is applied for $t_s = 5000$ and the final relaxation is applied for $t_r = 4000$.

First, the effect of $T_e$ on the system is studied in Figs. 4 and 8. In order to understand the behavior, shear stress is plotted against pressure and the ratio of the deviatoric-to-isotropic elastic strains is plotted against time. In the former Fig. 4, $T_e$ is practically zero and has no effect at all, whereas in the latter Fig. 8, the finite $T_e$ causes a reduced $T_g$ in steady state, as well as a much more rapid (exponential) relaxation to the static state (shorter magenta lines). Due the decreased $T_g$, the other state variables $\Delta$ and $u_s$ are increased, whereas their ratio is also decreased, see Eq. 70.

The effect of the new temperature production term with $f_g = 4.10^{-4}$ is then tested in Fig. 9 with otherwise the same settings as in case B. Only those cases that overshoot $g_e$ are affected. One of them, the lowest density case, is completely destabilized by the increase in $T_g$, while the other (second lowest density) remains above, but moves closer to $g_e$ and remains there for some longer time. This proofs that the production of $T_g$ due to the elastic instability allows to regularize the systems’ behavior by dynamic means, i.e., by generation of more $T_g$ keeps the system closer to the elastic instability. However, if too much $T_g$ is produced, this destabilizes the system and allows it to explore the plastic, collisional steady state with very large $T_g$ and – at the same time – small $u_s$ and $\Delta$.

Finally, we study the effect of different $f_g$ on a system at low density $\rho = 0.62$ using model 1 (case E) in Fig. 10, plotting again shear against normal stress in
Table 1 Summary of parameters used for the numerical solutions of GSH, where m. indicates the model used and dots replace values that are varied in this case.

<table>
<thead>
<tr>
<th>m.</th>
<th>$T_0$</th>
<th>$f_g$</th>
<th>$B$</th>
<th>$G$</th>
<th>$\lambda$</th>
<th>$\lambda_1$</th>
<th>$\alpha_1$</th>
<th>$R_{T0}$</th>
<th>$r$</th>
<th>$R_T$</th>
<th>$f_s$</th>
<th>$f_v$</th>
<th>$\eta_s$</th>
<th>$\chi$</th>
<th>$g_0$</th>
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<td>0.5</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>50</td>
<td>0.6</td>
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<td>4.10$^{-4}$</td>
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Fig. 8 Case B (model 0e with $T_0 = 2.10^{-4}$): Shear stress plotted against pressure (top) and deviatoric-to-isotropic elastic strain ratio plotted against time (bottom). The green lines (on the horizontal) represent the isotropic preparation, the magenta lines (overlapping), the final relaxation, with the big solid dots as theoretically predicted steady state $\sigma_{dev} = \mu^{(ss)}(\cdot)$. The dashed-dotted horizontal lines represent $\eta_s$ (upper) and $g^{(ss)}$ (lower), for finite $T_e$, while the dashed lines correspond to the critical state limits.

Fig. 9 Case C (model 0eg with $T_0 = 2.10^{-4}$ and $f_g = 4.10^{-4}$): Shear stress plotted against pressure (top) and deviatoric-to-isotropic elastic strain ratio plotted against time (bottom). The green lines (on the horizontal) represent the isotropic preparation, the magenta lines (overlapping), the final relaxation, with the big solid dots as theoretically predicted steady state $\sigma_{dev} = \mu^{(ss)}(\cdot)$. The dashed-dotted horizontal lines represent $\eta_s$ (upper) and $g^{(ss)}$ (lower), while the dashed lines correspond to the critical state limits.

The data are complemented by two more simulations, one with model 0, using the same density, and one with the same model 1 (with $f_g = 0$), but compressed up to density $\rho = 0.64$. The former is behaving very different, reaching the highest steady state level of $u_s/\Delta$ in steady state and also relaxing to a large value, $g > g_e$, because $f_g = 0$. The compression to higher density shows that this system, in steady state, is not reaching the elastic stability limit (upper diagonal) and, at the end of shear, is just relaxing deeper into the elastic cone (magenta line).

From the lower plot it is clear that all data reach their respective steady state and relax after shear is stopped. The four simulations with model 1 correspond to the four in-between curves, where the largest value of $f_g$ provides the curve that is closest to $g_e$, i.e., the temperature regularization succeeds to keep the system very close to the elastic stability limit, just by adding considerable temperature.

From the upper panel, we further learn that the model $g$ does not affect the system much if $f_g$ is close
to zero, but that the increased level of $T_p$, created by
the increasing $f_g$ values, keeps the system very close to
the stability limit (yellow curve) and allows the system
to relax to much smaller values stress, closely embrac-
ing the cricial state line $\mu_0$. In contrast to model 0, the
modified model 1 with large enough $f_g$ thus reaches a
very much relaxed final state, at rather small values of
stress, within the elastic stability cone.

This system thus has yielded when reaching the elas-
tic limit, $g_e$, there the temperature production kicks in,
proportional to $f_g$, and keeps the system close to $g_e$,
but pushing it towards the plastic equilibrium $\pi = 0$.
In the steady state the system is not reaching its de-
sired equilibrium, and also during relaxation it is not
just getting there, but rather jamming and becoming
elastic again.

5.2 Effect of dilatancy and dynamics

Next goal is to understand the behavior of the model at
constant density, with different dilatancy parameters,
$\alpha_1$, and the effects of the elastic dissipation parameter
$T_e$ and the temperature regularization $f_g$.

The initial preparation starts from an un-jammed
state at $\rho(0) = 0.58$, and is applied up to target density
$\rho = 0.65$, during $t_p = 1000$. From this point on, pure
shear is applied for $t_s = 5000$ and the final relaxation
is applied for $t_e = 4000$, like before.

The values of $\alpha_1$ are chosen such that a few of the
data remain within the elastic instability limit $u_s/\Delta <
g_e$, but a few overshoot, as can be seen in the lower
panels of Figs. 11, 12, and later the effect of $T_e$ and the temperature regularization $f_g$.

First, the effect of $f_g$ on the system is studied in
Figs. 11, 12, and later the effect of $T_e$ in Fig. 13. Again,
shear stress is plotted against pressure and the ratio
of the deviatoric-to-isotropic elastic strains is plotted

Fig. 10 Case E (model 1) with $T_p = 2.10^{-4}$ and different
$f_g = 0, 10^{-4}, 10^{-3}$, and $10^{-2}$: Shear stress plotted against
pressure (top) and deviatoric-to-isotropic elastic strain ratio
plotted against time (bottom). The single simulation with
model 0 corresponds to the uppermost curve in the lower
panel and the big solid dot is its theoretically predicted steady
state. The single simulation compressed towards larger den-
sity is the lowermost curve in the lower panel, and the right-
most curve in the upper panel. There, the green lines (on the
horizontal) represent the isotropic preparation, while the mag-
enta lines show the final relaxation after shear stops. The
slopes in the upper panel represent $\mu_0 = 2/g_e = 1$ and
$\mu = 0.5$ (to guide the eye), while the dashed-dotted hori-
zontal lines in the lower panel represent $g_e$ (lower) and $g^{(ss)}$
(upper, for model 0).

Fig. 11 Case D1 (model 0): Shear stress plotted against pres-
sure (top) and deviatoric-to-isotropic elastic strain ratio plotted
against time (bottom), for the same density, $\rho = 0.65$,
and different values of $\alpha_1 = 0.75, 1, 1.25, 1.5, 2$ (from top
to bottom). The green lines (on the horizontal) represent the
isotropic preparation, the curves the evolution during pure
shear up to the dots, representing the steady state solution,
$\sigma_{dev} = \mu_0^{(ss)} P$, see Eq. (78) while the magenta lines show
the final relaxation, with $T_e = 0$. The slopes in the top panel
correspond to $\mu_0^{(ss)} = 1$ and $\mu_c = 0.5$, to guide the eye, and
the dashed horizontal lines in the lower panel represent the
analytical values $g_e = 2$ and various $g^{(ss)}$, see Eq. (76).
Note that $T_e$ has an effect within and outside, whereas $f_g$ is only active outside the elastically stable regime.

6 Conclusion and Outlook

The focus of this paper was on yielding and unjamming/jamming of granular matter, which was inspired by the late Bob Behringer, to whom this work is dedicated. In an attempt to combine theoretical considerations with numerical/experimental observations on granular matter, the authors propose a minimalist macroscopic model to capture qualitatively all states of granular matter, and which even can be solved analytically in several special, limit cases.

The system considered was a representative volume element of granular matter, without gradients and no walls. The granular material was considered in fluid-like and solid-like states, as well as during continuous changes across the states as well as during and after the transition from elastically stable to instable, which is the novel contribution, since the latter states can be highly dynamic – something that is not possible, e.g.,
in standard elasto-plastic approaches or critical state
theory.

Based on the rather complex, but versatile granular
solid hydrodynamics (GSH), a much simplified quali-
tative model that includes un-jammed, fluid-like states
as well as jammed solid-like states (elastically stable)
was proposed and studied – analytically as well as nu-
merically. Various transitions and inter-
mediate states could be identified and better under-
stood in the framework of the simplestest GSH type
model, which has only three state-variables, density,
elastic strain (isotropic and deviatoric) and granular
temperature, unifying all the states of granular matter
we could imagine. In order to keep this universal mod-
eling attempts transparent, the model equations were
much simplified by making most parameters constant,
so that the structure of the model equations rather than
the constitutive assumptions could be tested.

This over-simplified model – even though not quan-
titatively calibrated, neither with experiments nor with
particle simulations – nevertheless, is capable of follow-
ning the granular system from very low (dilute granu-
lar gas) to very high densities (dense jammed granu-
lar solid), including various transients and transitions.
Furthermore, the model was generalized to include soft
particle phenomenology, as inspired by recent soft par-
ticle simulations, as well as a strictly non-thermal limit
removing the granular temperature), as well as per-
fectly plastic, elastic or intermediate states – involving
a critical state and an elastic instability, which was ac-
tually the main focus and reason to start this research.

The first mode of isotropic un-jamming appears
trivial; decompression of the system makes the den-
sity decrease and un-jamming takes place when the
elastic strain vanishes. However, the density at which
un-jamming takes place depends on the history of the
packing. Perturbations by tapping or over-compression
both can result in (un-)jamming densities considerably
larger than the lowest possible one, the random loose
packing density. The longer/stronger the system is per-
turbed, the larger the jamming density will be, but the
approach to this upper limit is realized very slowly.

Whether there are well defined random loose and ran-
don close packing densities, below/above which the sys-
tem cannot jam/un-jam anymore – or if not – is an
important open question: both limit densities are very
sensible to the protocol one uses to approach and real-
ize/measure them.

The second mode of un-jamming is by plastic
yielding, which involves irreversible deformations/re-
structuring of the solid granular matter, but does not
involve dynamics or granular temperature – at least
not in the classical picture. Plastic events occur with
a certain probability, see Ref. [53], which is larger the
closer the system is to un-jamming or the larger the
elastic shear strain (stress) is which was previously ac-
cumulated. This mode involves the more classical world
of elasto-plastic continuum mechanics and rheology for example see Refs. [119][40][60]. The evident lack of a
dynamic state variable is at the origin of many diffi-
culties with those elasto-plastic concepts, in particular
when the deformation rates become larger and larger.
Modern concepts like fluidity or non-local models have
been proposed during the last years to overcome this
problem [119][120][143].

The third mode of un-jamming is a transition oc-
curing via an elastic instability, i.e., the loss of convex-
ity, and then involves deformations of the solid granu-
lar matter that can occur without penalty, at the on-
set of concavity (elastic instability) or, are even acti-
ated/pushed by the external stresses (in the concave
regime). This mode is different from plastic yielding,
since it allows for dynamics (granular temperature) to
build up, grow, and eventually push back the system
into a mechanically stable elastic state before/while it
is dissipated. How much different – if at all – plastic and
elastic yielding really are has to be seen, and is subject
of current ongoing research.

Outlook: Many remaining challenges, besides the
quantitative calibration of the universal model for gran-
ular matter, involve the understanding of all the dif-
f erent mechanisms of relaxation, creation and destruc-
tion of energy in the elastic strain degrees of freedom
as well as the dynamic, kinetic, granular ones. Related
open questions are: What is the relaxation/evolution
dynamics of the state-variables below, above and during
un-jamming/jamming? What are the differences and
similarities of the driving forces/mechanisms? And, can
they all be combined in a single universal model as at-
tempted in this study?

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