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We study a two-dimensional granular gas of inelastic spheres subject to multiplicative driving proportional to a power $|v(\mathbf{x})|^\delta$ of the local particle velocity $v(\mathbf{x})$. The steady state properties of the model are examined for different values of δ , and compared with the homogeneous case $\delta = 0$. A driving linearly proportional to $v(\mathbf{x})$ seems to reproduce some experimental observations which could not be reproduced by a homogeneous driving. Furthermore, we obtain that the system can be homogenized even for strong dissipation, if a driving inversely proportional to the velocity is used.

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Granular materials are a fascinating non-linear, dissipative, non-equilibrium system [1, 2], which are of interest, first due to their practical importance and second because of the challenges they represent theoretically and numerically, but experimentally as well. Granular media are collections of macroscopic particles with rough surfaces and dissipative, frictional interactions. For the sake of simplicity, we restrict ourselves to the smooth situation here, and use the inelastic hard sphere (IHS) model [3–6]. For more realistic interaction models see Refs. [2, 7] and the references therein.

One of the outstanding effects is the so called clustering, a self-stabilized density instability due to dissipation [3, 6] where large, dense collections of particles coexist with almost empty areas. Clustering occurs in initially homogeneous systems [1, 3, 6, 8] and should not be confused with the so-called “inelastic collapse” [3], the divergence of the collision rate, which is inherent to the frequently used hard (rigid) sphere model [3, 6, 7]. Freely cooling systems – mainly examined numerically and theoretically [3] – are almost impossible to realize experimentally. Only recently, laboratory experiments were performed, where clustering could be examined in driven systems [9–11], where also kinetic theory approaches complemented by numerical simulations have proven to be successful [4, 5, 7, 12–15].

The driving of a granular material can be realized by moving walls [2] which lead to local heating, or the system can alternatively be driven by a global homogeneous, random energy source in different variations [4, 5, 14, 15]. The latter type of energy input does not exactly resemble the experimental situation, where a two-dimensional (2D) horizontal layer of spheres is agitated by vertical vibrations of the bottom surface and the horizontal degrees of freedom are indirectly agitated due to the different vertical jump heights of the colliding particles [10, 12]. Thus, the choice of the driving term to put into a theory for dissipative systems is an open problem and we expect that it depends on the nature of the driving (vibrating wall, airflow, Brownian noise, etc.). In order to

find possible candidates for a realistic energy input, the IHS model with an inhomogeneous, *multiplicative driving* is examined in the following. The driving is proportional to the local velocity $|v(\mathbf{x})|^\delta$, with a given power δ which can take both positive and negative values. The classical homogeneous driving with $\delta = 0$ is contained in our approach as a special case. A positive power leads to weak energy input for slow particles, e.g. those moving collectively inside a cluster [10, 12].

In this letter a system of N three-dimensional spheres with radius a and mass m is considered, interacting via a hard-core potential and confined to a 2D plane of linear extension L , with periodic boundary conditions. The degrees of freedom are the positions $\mathbf{r}_i(t)$ and the translational velocities $\mathbf{v}_i(t)$ for each sphere numbered by $i = 1, \dots, N$. The dissipation at a collision is quantified by a constant normal restitution r . From the momentum conservation law we can derive the change of linear momentum $-(m/2)(1+r)\mathbf{v}_c^{(n)}$ of particle i which collides with particle j , with the normal relative velocity $\mathbf{v}_c^{(n)} = [(\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{n}]\mathbf{n}$, and the unit vector in normal direction $\mathbf{n} = (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|$. The system is agitated each time interval $\Delta t = f_{\text{dr}}^{-1}$, with a driving rate f_{dr} according to the frequency of the bottom. For homogeneous driving, it is customary to assume $f_{\text{dr}} \gg \omega$, where ω is the collision frequency of the granular gas [4, 5]. This, however, does not correspond to the experiments, where f_{dr} can be rather small – we will use driving frequencies around 100 s^{-1} , comparable to experimental situations [10]. Numerical checks with strongly different values of f_{dr} lead to a similar behavior of the system even for driving frequencies lower than, but of the same order as ω , provided that a stationary state is reached. The velocity of particle i is modified at each time of agitation t so that

$$\begin{aligned} v_i^{\prime x}(t) &= v_i^x(t) + r_i^x |v_i(t)|^\delta v_r^{1-\delta} \\ v_i^{\prime y}(t) &= v_i^y(t) + r_i^y |v_i(t)|^\delta v_r^{1-\delta}, \end{aligned} \quad (1)$$

where the prime on the left hand side indicates the value after the driving event. v_r is a reference velocity (in

this study we use $v_r = 1 \text{ ms}^{-1}$) which allows to define the dimensionless translational particle temperature $T = E/(NT_r)$, with $E = (m/2) \sum_{i=1}^N \mathbf{v}_i^2$ and the reference temperature $T_r = mv_r^2$. The variance of the uncorrelated Gaussian random numbers r_i^x and r_i^y (with zero mean) can now be interpreted as a dimensionless driving temperature T_{dr} . The stochastic driving rule in Eq. (1) leads thus to an average rate of change of temperature

$$\Delta T/\Delta t = H_{\text{dr}} T^\delta \quad \text{with} \quad H_{\text{dr}} = f_{\text{dr}} T_{\text{dr}}. \quad (2)$$

The time evolution equation of T was derived for the case of a freely cooling granular gas by means of a pseudo-Liouville operator formalism [6, 8]. We adopt their nomenclature and account for the driving by adding Eq. (2) to the mean field (MF) equation for the translational degree of freedom

$$\frac{d}{dt} T(t) = -G_r A T^{3/2} + H_{\text{dr}} T^\delta. \quad (3)$$

For our case of a homogeneous monolayer of smooth spheres, one has $G_r = 8an\sqrt{\pi T_r/mg(\nu)}$, and $A = \frac{1-r^2}{4}$, with the number density $n = N/V$, the pair correlation function at contact $g(\nu)$, and the area fraction $\nu = N\pi a^2/V$ covered by particles [6–8]. For the two-dimensional layer of spheres we use the approximation $g(\nu) = (1 - 7\nu/16)/(1 - \nu)^2$ [16]. For $\delta = 0$ the driving is homogeneous and independent of the local granular temperature (or velocity). In the case $\delta \neq 0$ the driving is a function of T [17]. Imposing $\frac{d}{dt} T(t) = 0$ one gets from Eq. (3) the MF temperature in the steady state

$$T^{\text{mf}} = \left(\frac{H_{\text{dr}}}{G_r A} \right)^{\frac{2}{3-2\delta}}, \quad (4)$$

the generalization of the Enskog equilibrium solution for a driven granular gas [4, 5]. The scaling exponent of T^{mf} in Eq. 4 is $2/3$ for $\delta = 0$, while it is 2 for $\delta = 1$. For $\delta \geq 3/2$ the MF theory does not admit a stable equilibrium state [7, 18].

The final approach to the steady state can be obtained by linearizing Eq. (3) around T^{mf} , what leads to an exponentially fast approach to equilibrium with the power $-[3A\omega + \delta H_{\text{dr}}(T^{\text{mf}})^{\delta-1}]t$, where $\omega = G_r \sqrt{T^{\text{mf}}}/2$ is the Enskog collision frequency for elastic particles with temperature T^{mf} . By inserting Eq. (4) in the expression for ω , one can express the characteristic relaxation time $t_{\text{relax}} = [\dots]^{-1}$ of $T(t)$ as a function of the model parameters, which reduces for $\delta = 1$ to $t_{\text{relax}}^{-1} = (5/2)H_{\text{dr}}$. Thus, for $\delta = 1$ the characteristic time for the evolution of T towards its equilibrium value does *not* depend on A which contains all the information about the inelasticity. This characteristics is confirmed by numerical simulations.

Most of our event driven (ED) molecular dynamics simulations, see [6–8] for details, with the driving specified in Eq. (1), are first equilibrated without driving and

with elastic interactions ($r = 1$), until the velocity distribution is close to a Maxwellian. Then, dissipation and driving are switched on. However, we checked that the steady state does not depend on the initial conditions. The simulations are performed at fixed volume fraction $\nu = 0.34$ with $N = 1089$ ($f_{\text{dr}} = 133 \text{ s}^{-1}$) or $N = 11025$ ($f_{\text{dr}} = 67 \text{ s}^{-1}$), and different values of r . In our simulations we have chosen $a = 10^{-3} \text{ m}$ and $H_{\text{dr}} = 1.0 \text{ s}^{-1}$ so that, for example, $G_r = 8\nu v_r / \sqrt{\pi} a g(\nu) = 3.1 \times 10^3 \text{ s}^{-1}$, $T^{\text{mf}} = 0.0358$ and thus $\omega = 2.9 \times 10^2 \text{ s}^{-1}$, if $r = 0.90$ is used. With these typical values and a homogeneous driving ($\delta = 0$), the model of a driven granular gas is very close to a homogeneous state; no clusters are observed and the velocity distribution is almost Maxwellian [7]. This is in contradiction to the experimental findings [10], and suggests that the correct representation of the driving in those experiments is *not the homogeneous white noise* usually implemented.

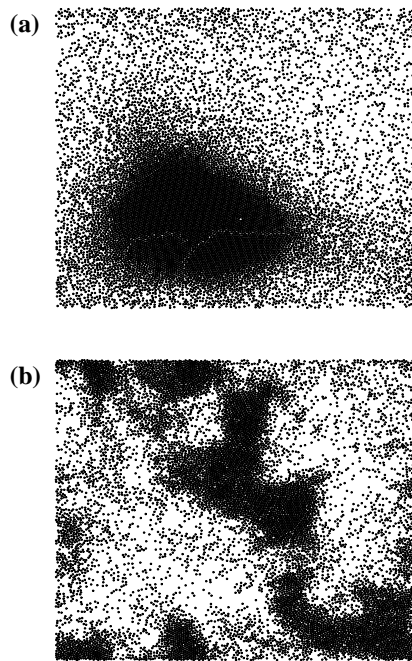


FIG. 1. Snapshots of the particle distribution in the steady state for a system of $N = 11025$ particles, $\delta = 1$, $\nu = 0.34$, and $H_{\text{dr}} = 1.0 \text{ s}^{-1}$, with $r = 0.97$ (a) and $r = 0.6$ (b).

In Fig. 1 snapshots of the system's steady state are shown for $r = 0.97$ (a) and $r = 0.6$ (b), with $\delta = 1$. Different regimes are observed: A homogeneous state exists for very weak dissipation ($r = 0.999$, not shown here), whereas dense, persistent clusters with a crystalline structure, domain boundaries, and vacancies are found for higher dissipation ($r = 0.97$). The region between these clusters appears rather homogeneous and dilute (gas-like), very similar to the structures observed in experiments [10]. For higher dissipation ($r = 0.6$) the clusters appear less symmetric and are smaller.

Note that the clustering in the case $\delta = 1$ is qualitatively different from the case of homogeneous driving, and it appears already for quite high values of r (see Fig. 1). Homogeneous driving, in fact, leads to transient clusters, i.e. they appear and disappear continuously, while in the experiments [10] and in the case of multiplicative driving, the individual clusters are in equilibrium with a gas phase and are stable for rather long times. Simulations with negative δ , (we used $\delta = -0.5$ and $\delta = -0.25$) give a behavior qualitatively similar to the case of homogeneous driving ($\delta = 0$). For $\delta = 1$, the particles inside the cluster, with rather small relative velocities, are much less agitated than the particles in the surrounding gas phase, so that a cluster is stable. For $\delta \leq 0$, the particles in the cluster are driven comparatively strong, what leads to less stable, dynamic clusters.

In Fig. 2 we plot the ratio between the numerical results for long times T^{eq} and the theoretical equilibrium temperatures T^{mf} as function of r for different δ . The agreement of the simulations with the MF prediction is optimal for $r \rightarrow 1$. For $\delta < 0$, the range of agreement extends to much smaller r values, i.e. to stronger dissipation, as in the case of $\delta = 1$ and even in the case $\delta = 0$.

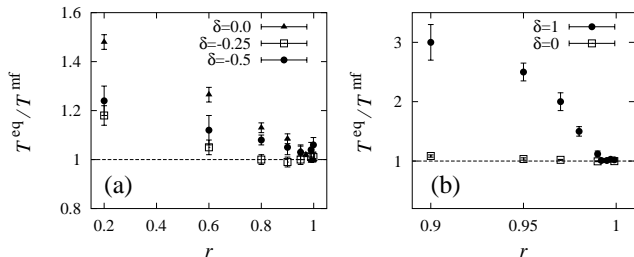


FIG. 2. Rescaled translational temperature $T^{\text{eq}}/T^{\text{mf}}$ plotted against the restitution coefficient r for $N = 11025$ and different values of δ as given in the insets. Note the different axis scaling in (a) and (b).

We have also performed simulations with $\delta = -1$, and found that the driving is very singular in the low velocity limit, i.e. an excessive amount of energy is given to slow particles, creating an inhomogeneous state.

In the following, we focus on the steady state velocity distribution. Therefore, we performed a large series of simulations for different values of r , with $\nu = 0.34$, $\delta = 1$, and both $N = 1089$ and $N = 11025$. For each value of r we performed about 10^3 simulations for $N = 1089$ and about 2×10^2 simulations for $N = 11025$, each with different initial configurations. The distribution of velocity is symmetric and isotropic so that we present only the data for the x component. For data analysis, a Gaussian with the width obtained from the numerical simulations can be superposed to the velocity distribution function $f(v_x)$ in order to visualize the differences. Furthermore, as a more quantitative approach, a three parameter fit

function $f_\alpha(v_x) = f_0 \exp(-B|v_x - \langle v_x \rangle|^\alpha)$ is used to estimate the exponent α of the tail of the distribution. The results are reported in Fig. 3, where a representative velocity distribution is shown in the inset. The main outcome of our simulations is that the velocity distribution is not Gaussian for high inelasticity and that the exponent α in the stretched exponential depends on the normal restitution r . The non-Gaussian behavior is limited to the tail for large r , but seems to extend over the whole range of velocities if r is small enough.

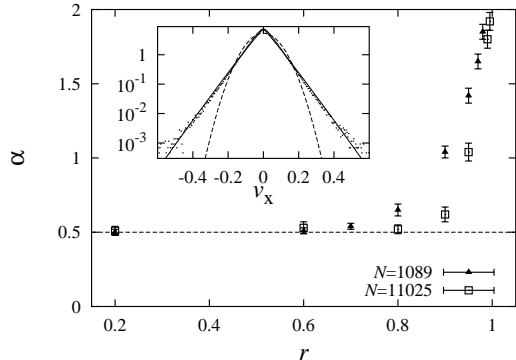


FIG. 3. Plot of the exponent α as a function of r , for system size $N = 1089$ (solid triangles) and $N = 11025$ (open squares). In the inset, the dots give the averaged, normalized velocity distribution from simulations with $N = 1089$ and $r = 0.9$, the solid line is a fit according to $f_\alpha(v_x)$ with $f_0 = 7.6$, $B = 18.9$ and $\alpha = 1.05(9)$, and the dashed line is a Gaussian with the same standard deviation as the data.

In Fig. 3, the exponent α varies continuously between 0.5 and 2 and, in particular, for $N = 1089$ and $r = 0.9$ we get $\alpha = 1.05(9)$, while for $N = 11025$ and $r = 0.95$ we observe $\alpha = 1.02(6)$. Such an almost exponential velocity distribution has recently been observed in experiments on a vibrated monolayer of spheres, together with the cluster structure discussed above [10]. The size dependency of our results, seems reasonable, since also in freely cooling systems the value of r below which the inhomogeneity and the clustering become important grows with the system size [3, 6, 7]. Simulations for $\delta = -0.25$ and -0.5 , show that the velocity distribution is very near to a Gaussian even for strong dissipation.

In order to study the shape of the velocity distribution function analytically, we extend the approach used by van Noije et al. [5] in the special case $\delta = 0$ to arbitrary δ . The Enskog-Boltzmann equation for the freely cooling gas of spheres, with the standard collision integral $I(f, f)$, has to be extended by the multiplicative driving term proportional to the particle velocity. The Enskog-Boltzmann equation, see Eq. (26) in [5], corrected by a Fokker-Planck diffusion term [17], is

$$\frac{\partial}{\partial t} f(\mathbf{v}_1, t) = g(2a)I(f, f) + \frac{H_{\text{dr}}}{2m} \frac{\partial^2}{\partial v_1^2} [v_1^{2\delta} f(\mathbf{v}_1, t)]. \quad (5)$$

Considering the stationary limit $\partial f(\mathbf{v}_1, t)/\partial t = 0$, and introducing a scaled distribution function, with the dimensionless velocity $c = v/v_0$ and the mean thermal velocity v_0 , one obtains a dimensionless evolution equation for the scaled $f(\mathbf{v})$, as in [5]. Multiplied by c_1^p and integrated over \mathbf{c}_1 , this leads to a set of equations which couple the moment $\langle c^{p-2+2\delta} \rangle$ to the p -th moment μ_p of the collision term. In the special case $\delta = 1$ and $p = 2$, one obtains $v_0 = H_{\text{dr}}/(g(v)\mu_2an)$. For $\delta \neq 1$ and $\delta \neq 0$ the MF thermal velocity can be obtained by assuming $\langle c^{2\delta} \rangle = [\langle c^2 \rangle]^\delta = v_0^{2\delta}$ [19].

For high c , one has $\tilde{I}(\tilde{f}, \tilde{f}) \approx -\beta_1 c_1 \tilde{f}(c_1)$, with $\beta_1 = \pi^{1/2}$ in 2D and the equation for $f(c)$ becomes

$$-\beta_1 c \tilde{f}(c) + \frac{\mu_2}{2d} \left(\frac{d^2}{dc^2} + \frac{d-1}{c} \frac{d}{dc} \right) [c^{2\delta} \tilde{f}(c)] = 0, \quad (6)$$

Inserting a solution of the form $\tilde{f}(c) \propto \exp(-Bc^\alpha)$, we obtain the large c solution $\alpha = \frac{3-2\delta}{2}$ and $B = \frac{2}{3-2\delta} \sqrt{2d\beta_1/\mu_2}$. In particular, for $\delta = 1$ we find $\alpha = 1/2$ what corresponds to the exponent we found in the low r limit, see Fig. 3. Our explanation is that there is a crossover velocity c^* above which the tail with exponent $1/2$ appears, which decreases as r decreases. Such a tail is not observed numerically for high r values, since c^* is outside the velocity range we can examine. Simulations with $r = 0.2$, $N = 1089$, and $\delta = -0.5, -0.25$, and 0.5 give respectively $\alpha = 2.10(9)$, $1.90(9)$, and $1.10(7)$, quite near to the values $\alpha = 2, 7/4$, and 1 , given by the large c analysis. Since the driving depends on the velocity, it acts on all the velocity scales, affecting the shape of $f(v)$ in a more complex way than in the case of homogeneous driving. Moreover, when inhomogeneities are present, the driving will depend on the local thermal velocity $v_0(\mathbf{x})$ and on the local density $n(\mathbf{x})$, while homogeneous driving is *identical everywhere*. Since inhomogeneities are generated by the dissipation, a *feed-back* between driving and dissipation could be at the origin of the non-universal behavior of $f(v)$.

Future research includes a MF and clustering study for a wider parameter range, including the case of rough particles. Both a more detailed analysis of $f(v)$ [7] and a quantitative analysis of the clusters [20] are in progress. A microscopic justification for the multiplicative driving is still lacking and requires more detailed experimental or three-dimensional simulation [12] studies. However, we suggest a possible experimental check of our ideas. From MF theory the scaling of the equilibrium temperature of the vertically vibrated gas with the area fraction ν is given by $T^{\text{mf}} \propto [\nu g(\nu)]^{-\frac{2}{3-2\delta}}$. Experimental measurement of the equilibrium temperature of the vertically vibrated monolayer, for different values of ν , could allow to estimate the value of δ and to verify the hypothesis of

a multiplicative effective driving.

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