the granular solid

Jacco Snoeijer (Twente): force networks

Brian Tighe (Leiden): jamming

the granular solid

an old problem...

a wise man who built his house on the rock

a foolish man who built his house on the sand...
and the house fell

(Matthew 7:24-27)

an old problem...

a wise man who built his house on the rock

a foolish man who built his house on the sand...

and the house fell

(Matthew 7:24-27)

soil mechanics

continuum (large scale) models:

stress and strength

force propagation

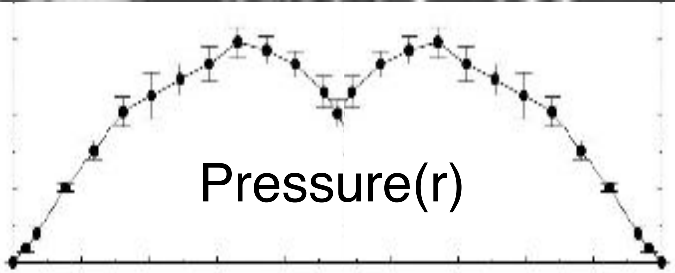
hopper

Pressure(r) ?

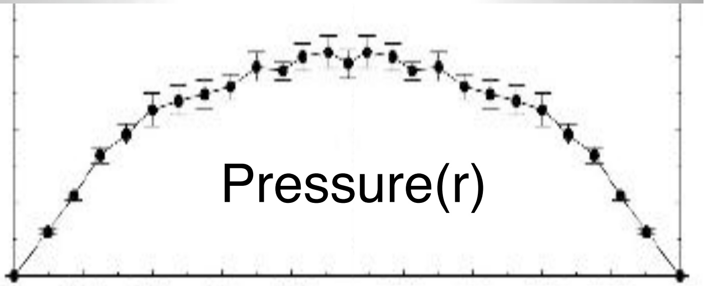
r →

force propagation

hopper



homogeneous ‘rain’



L. Vanel *et al.,* PRE **60**, R5040 (’99)

more recent papers: Atman et al. 2005, Mullin 2007

this lecture...

- some issues for packings
- force networks:
  - experimental motivation
  - some theoretical ideas

general question: can we understand <u>macroscopic</u> from <u>microscopic</u> behavior?

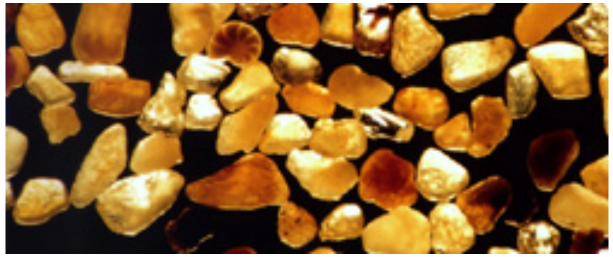
question:

what grain-level parameters are important for the properties of a granular solid?

parameters

grain properties:

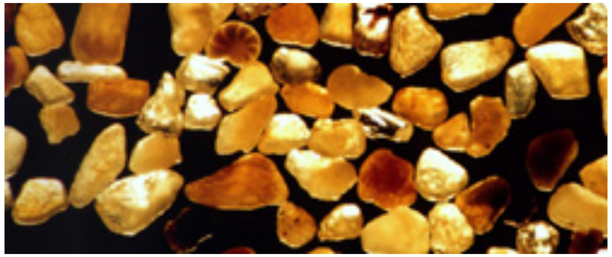
shape, sizes, friction, hardness, etc.



parameters

grain properties:

shape, sizes, friction, hardness, etc.



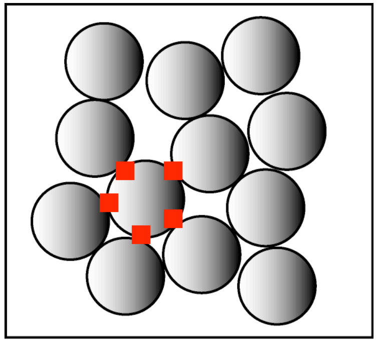
packing properties - local geometry:

of contacts

contact distribution

packing fraction

correlations, ... ???



parameters

packing properties - local geometry:

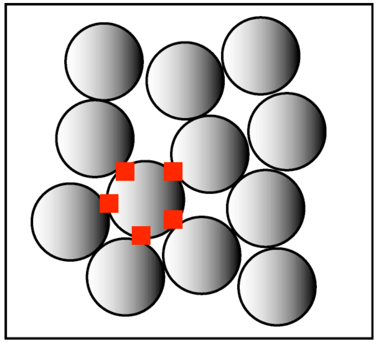
of contacts

contact distribution

packing fraction

correlations, ... ???

models: often simple spheres



parameters

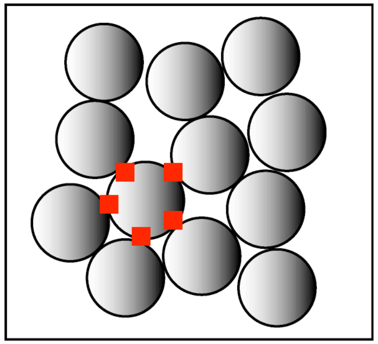
packing properties - local geometry:

of contacts

contact distribution

packing fraction

correlations, ... ???

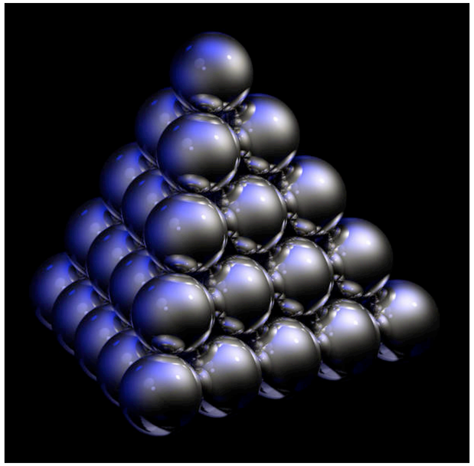


models: often simple spheres

I will mainly discuss frictionless spheres

close packing of spheres

Keppler conjecture 1611: fcc highest density



$$\phi = \frac{\pi}{\sqrt{18}} \approx 0.74048$$

Gauss 1831: proof for regular lattices

Hales 1998: proof for disordered packs by computer algebra

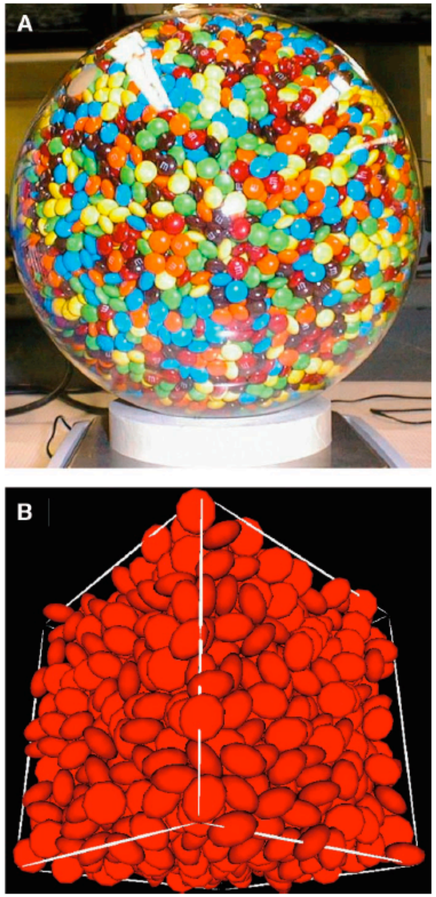
random close packing

disordered pack of spheres:

typically $\phi \approx 0.64$

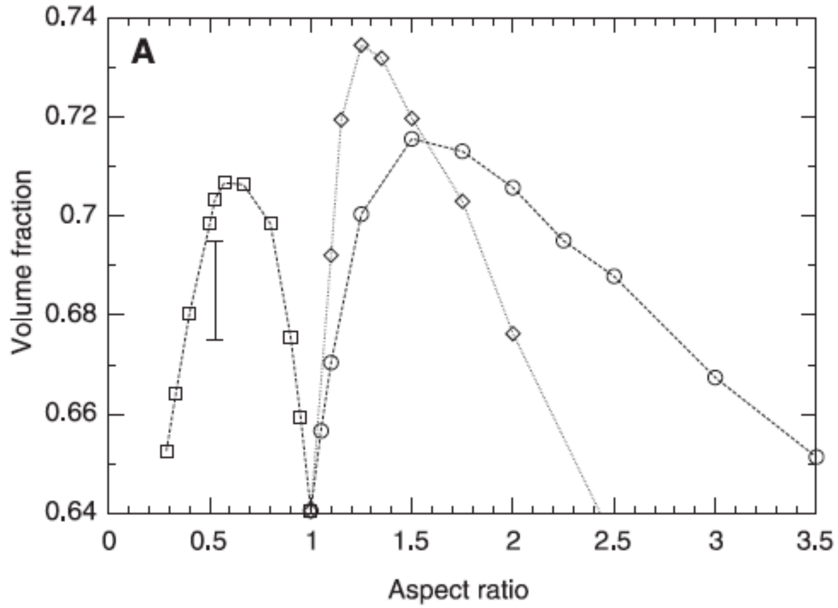
spheroids pack more densily:

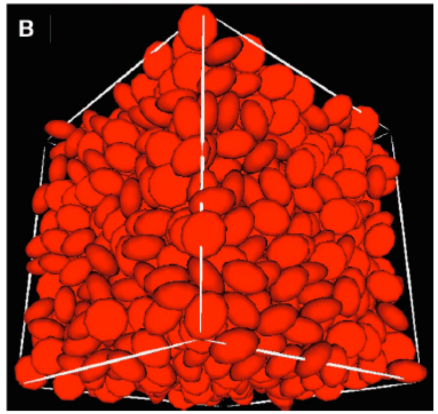
up to $\phi \approx 0.72$



Donev et al Science 2004
(Williams & Philipse PRE 2003)

random close packing





spheroids pack more densily:

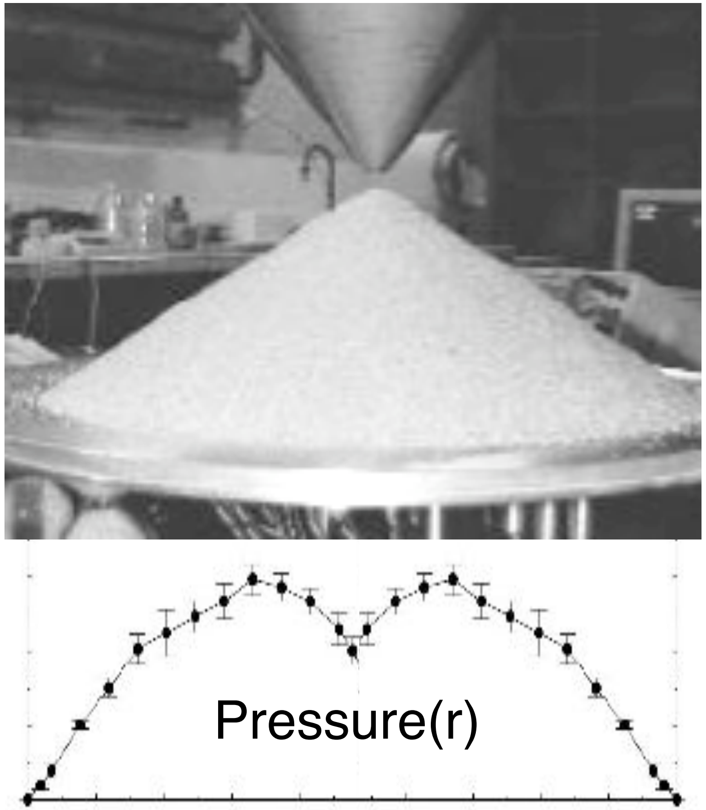
up to $\phi \approx 0.72$

Donev et al Science 2004
(Williams & Philipse PRE 2003)

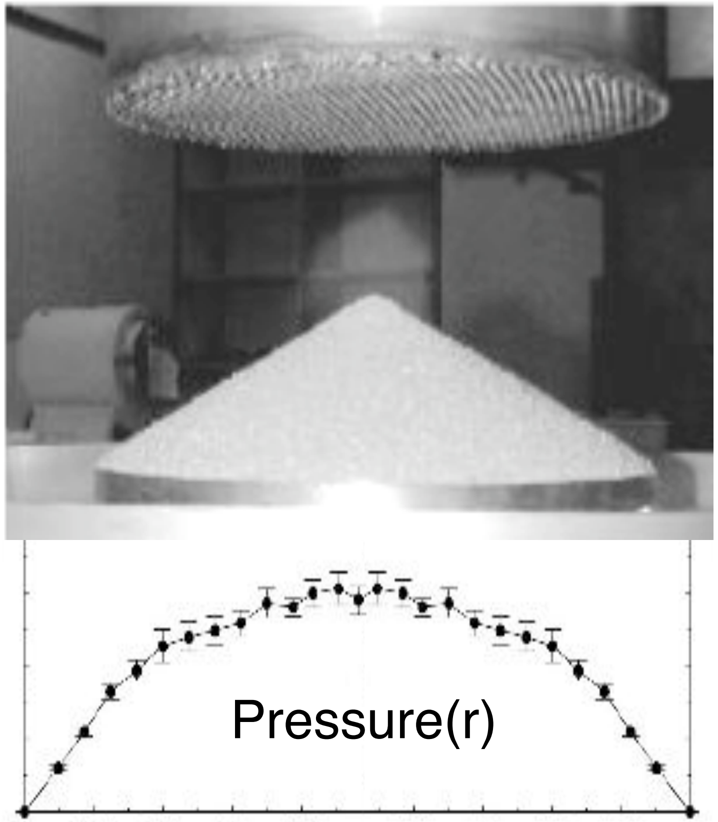
forces

forces

hopper

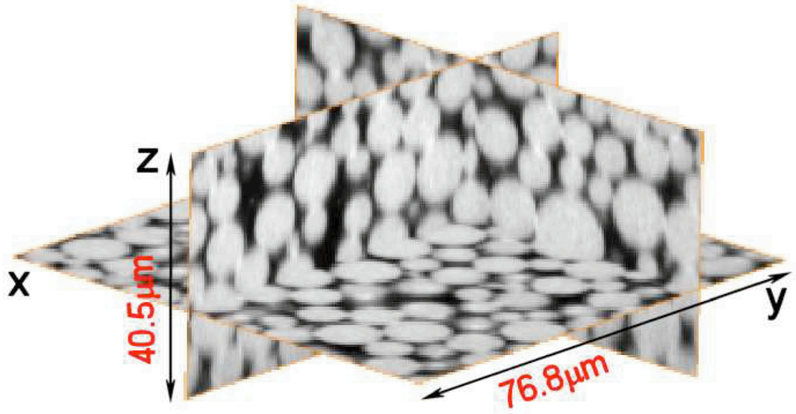


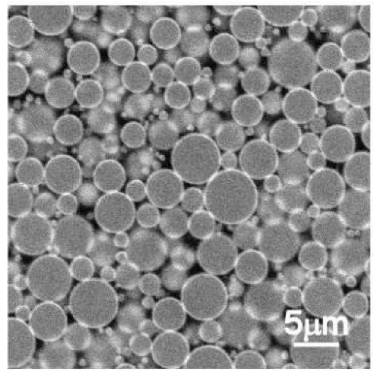
homogeneous ‘rain’



looking inside: emulsions

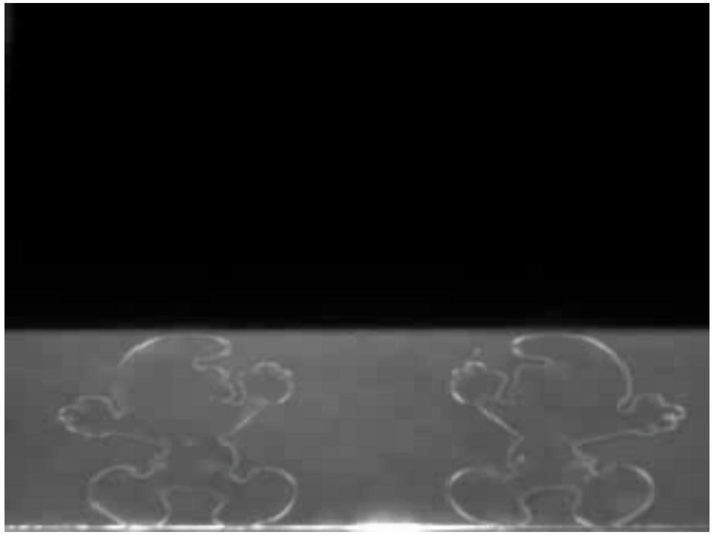
packing of oil droplets dispersed in liquid

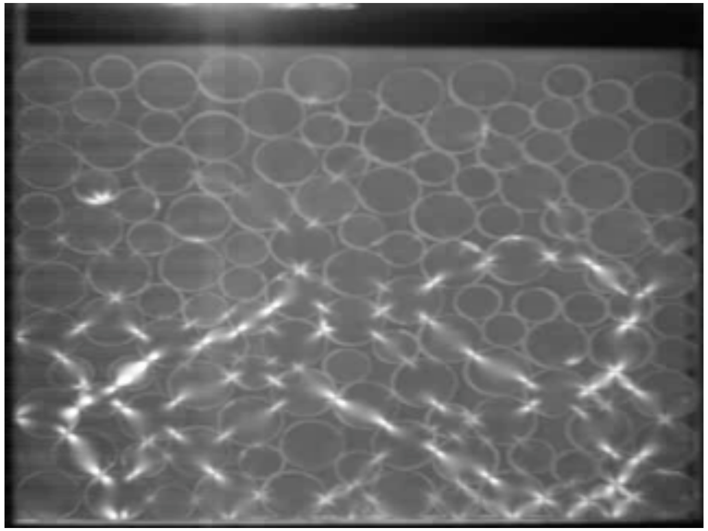




Brujic et al., Phys. A 2003
Zhou et al., Science 2006

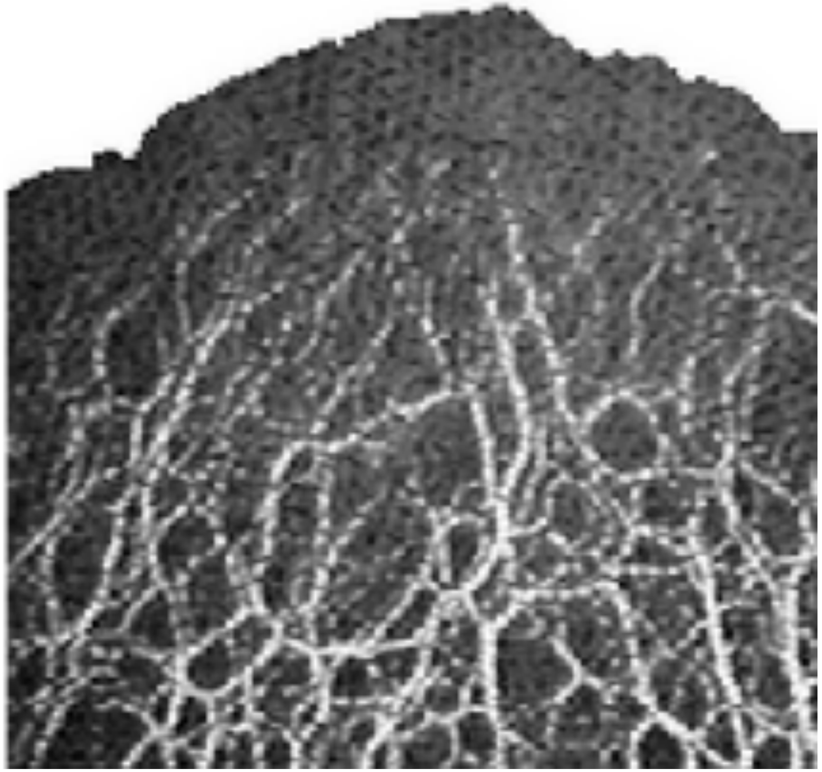
photo elastics: stress





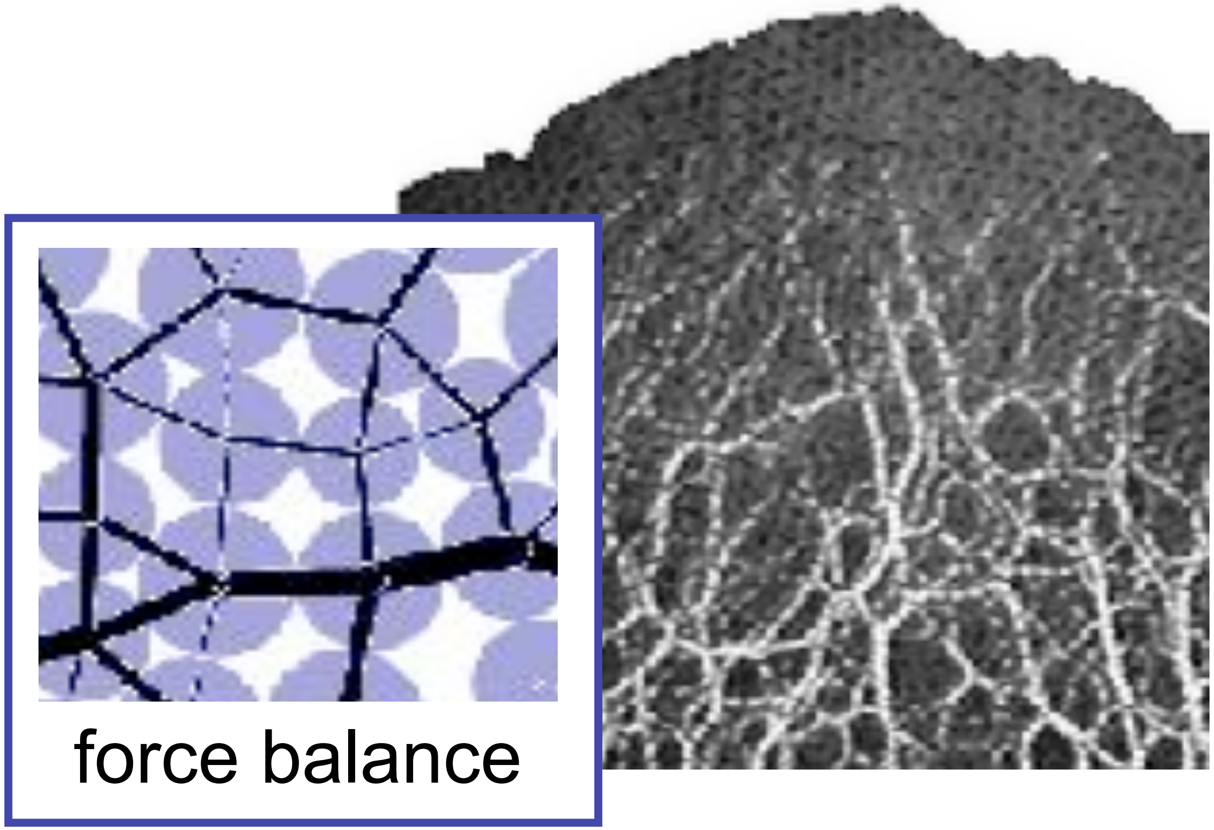
thanks to Martin van Hecke

force networks

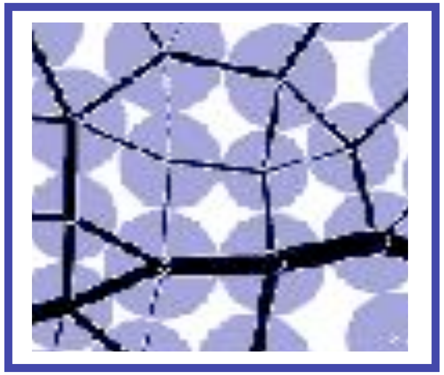


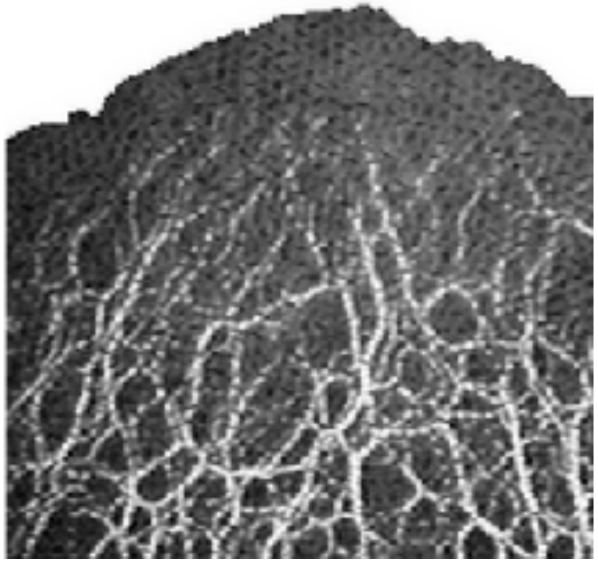
R.P. Behringer *et al.*

force networks



force networks

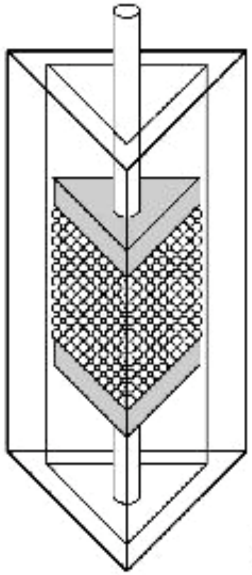




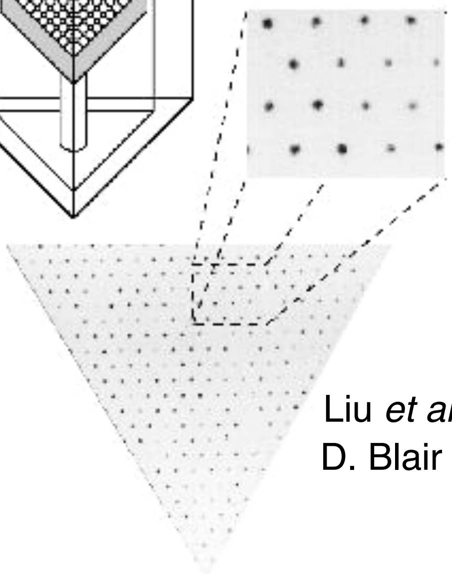
1. inhomogeneity -> force statistics?
2. structure: ‘force propagation’
   -> stress dip

force statistics

Carbon paper: P(F)



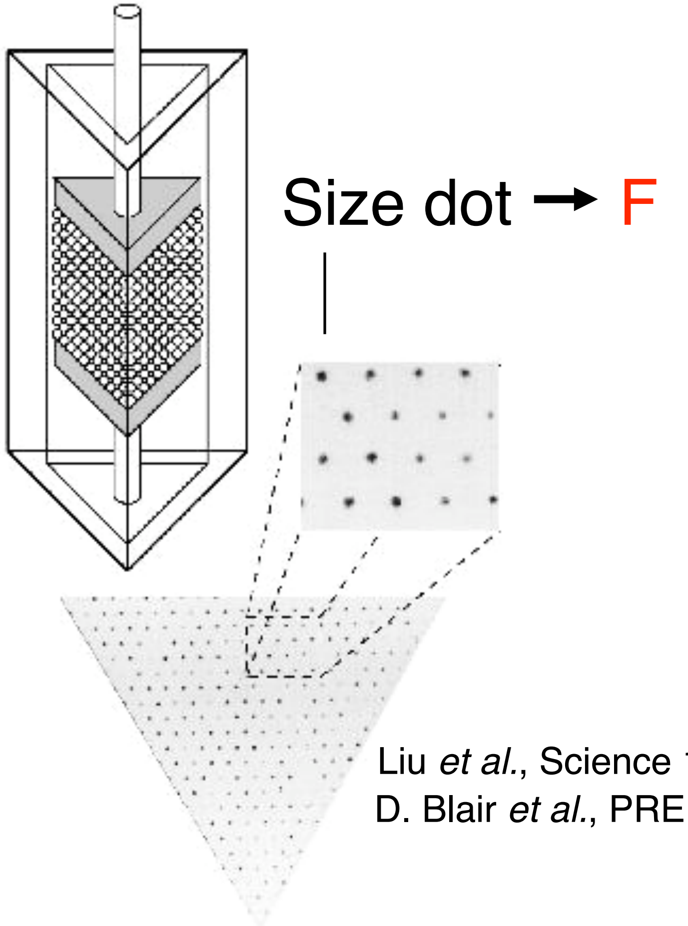
Size dot → F



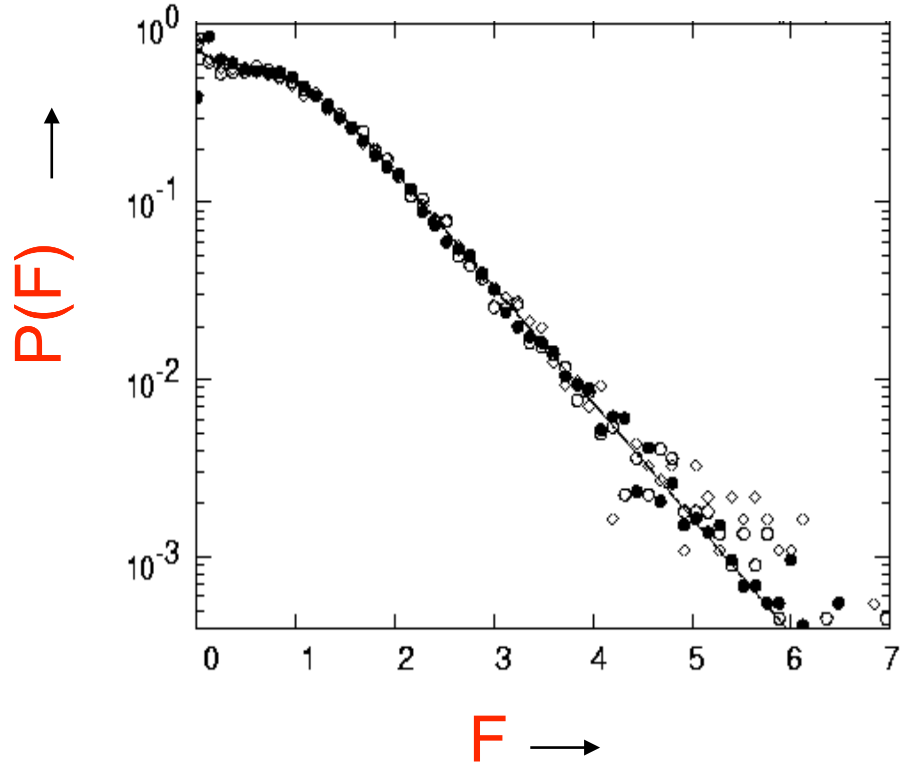
Liu *et al.*, Science 1995
D. Blair *et al.*, PRE 2001

force statistics

Carbon paper: P(F)



Liu *et al.*, Science 1995
D. Blair *et al.*, PRE 2001



force statistics

carbon paper

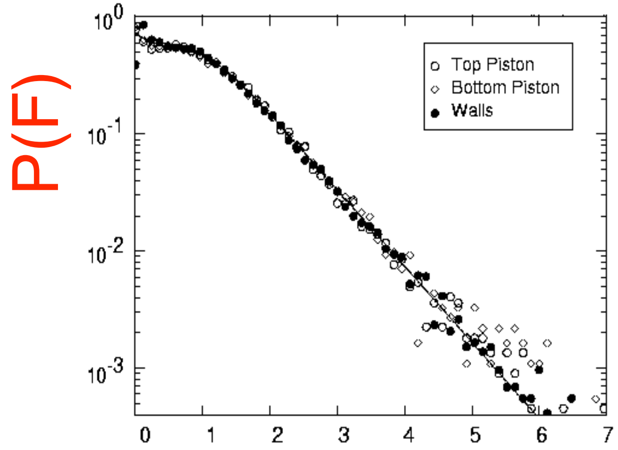
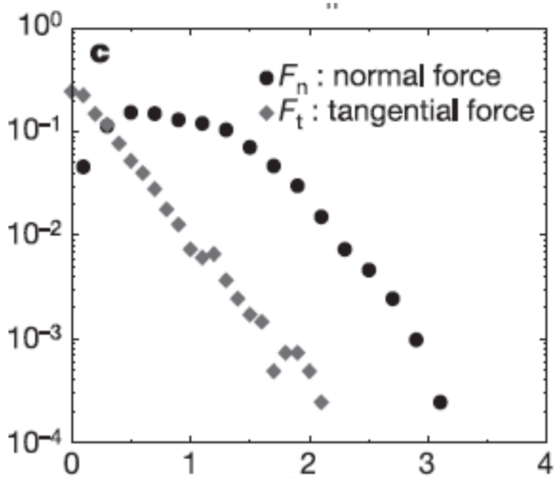


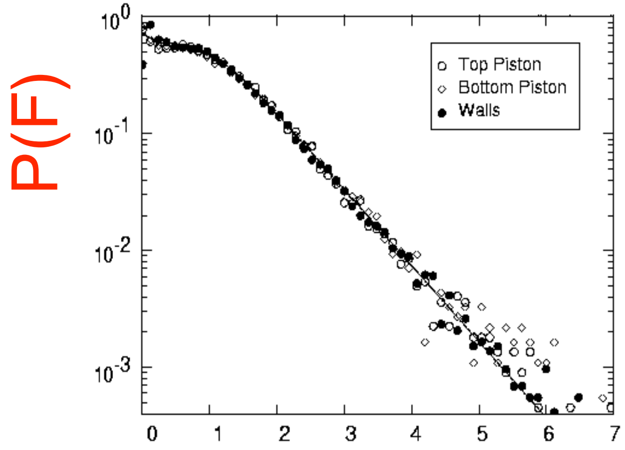
photo-elastic grains



Majmudar & Behringer, Nature 2005

large forces exist... but are exponentially rare

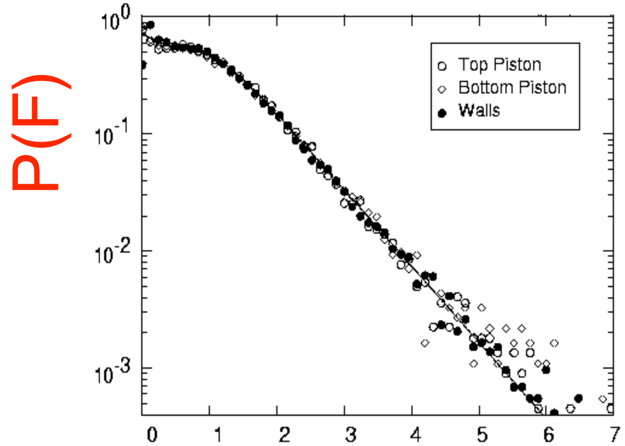
popular idea:



analogy with equilibrium ensembles, Boltzmann?

forces play the role of energies?

popular idea:



analogy with equilibrium ensembles, Boltzmann?

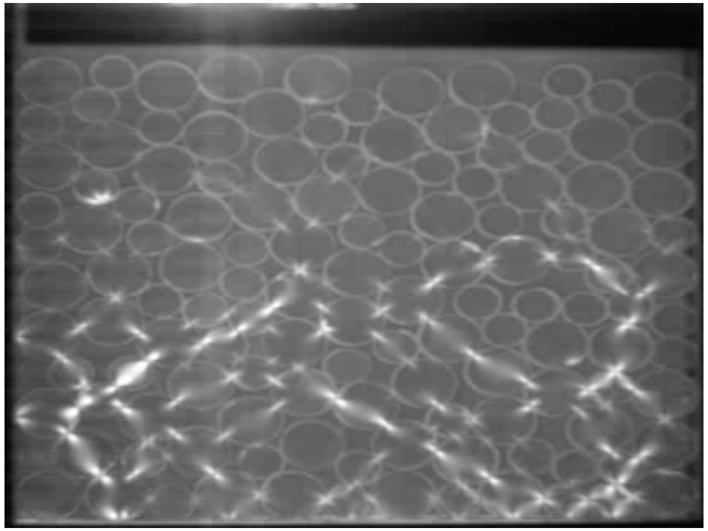
forces play the role of energies?

question: what is origin of $p \sim e^{-E/kT}$ in the canonical ensemble?

conserved quantity

imposing a pressure

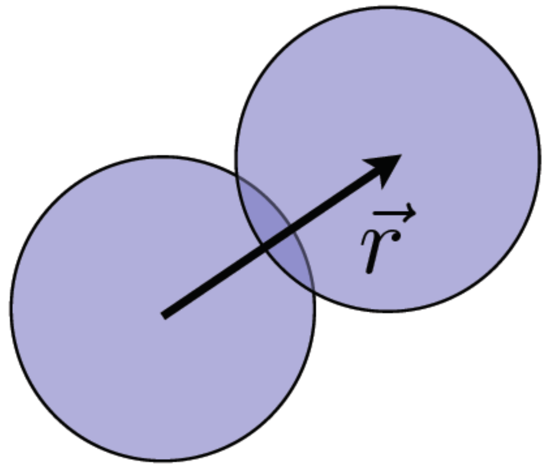
= imposing <F>



conserved quantity

imposing a pressure

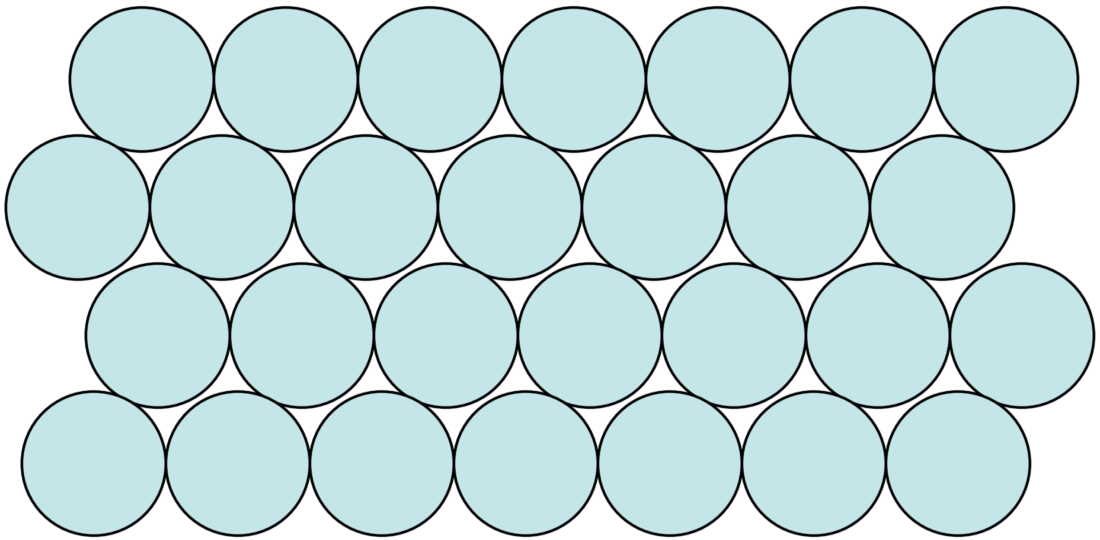
= imposing <F>



$$p = \frac{1}{V}\sum_{ij} r_{ij} f_{ij} \sim \frac{\langle r \rangle}{V}\sum_{ij} f_{ij}$$

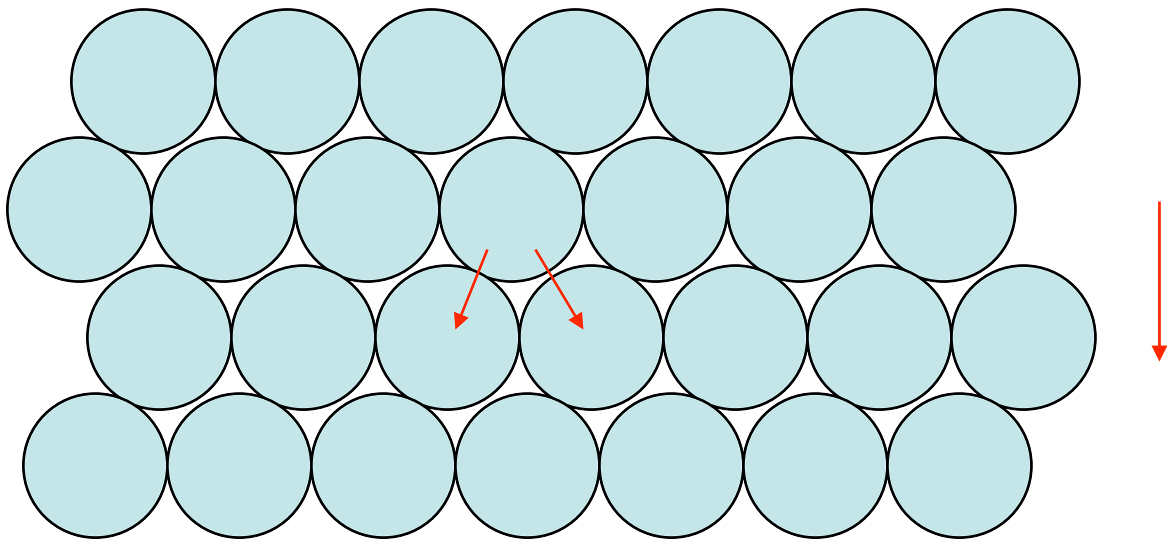
a simple model

- assume the balls are on a regular lattice

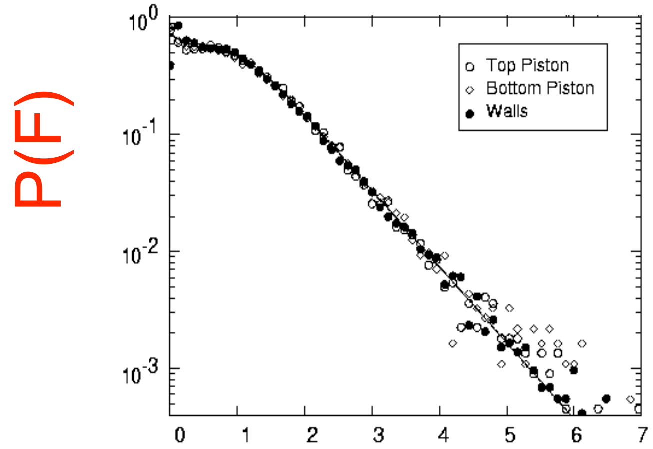


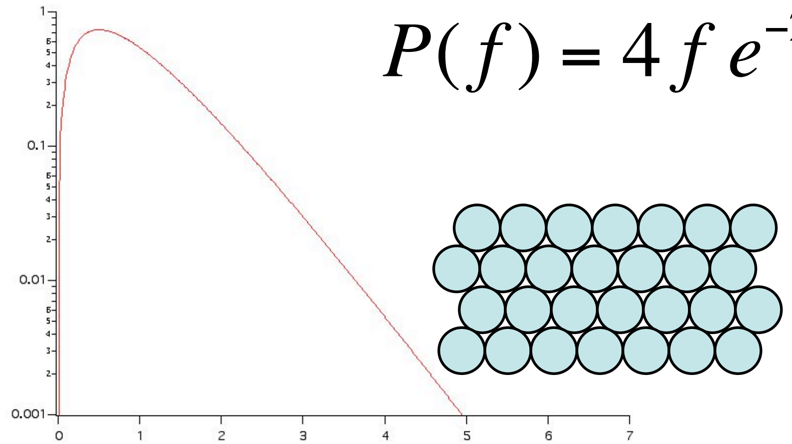
a simple model

- assume the balls are on a regular lattice
- assume forces are randomly 'transmitted' downwards from layer-to-layer



q-model

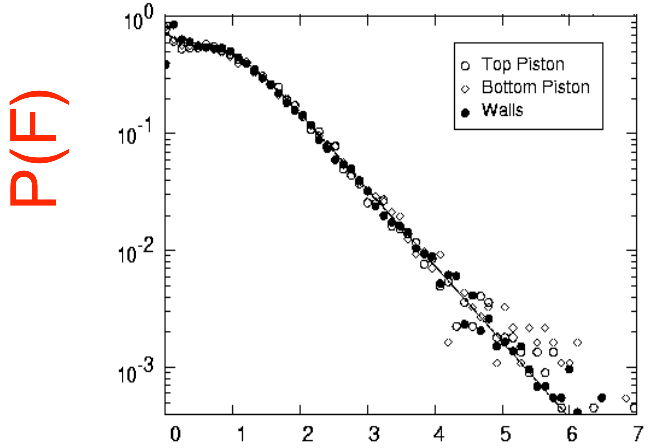


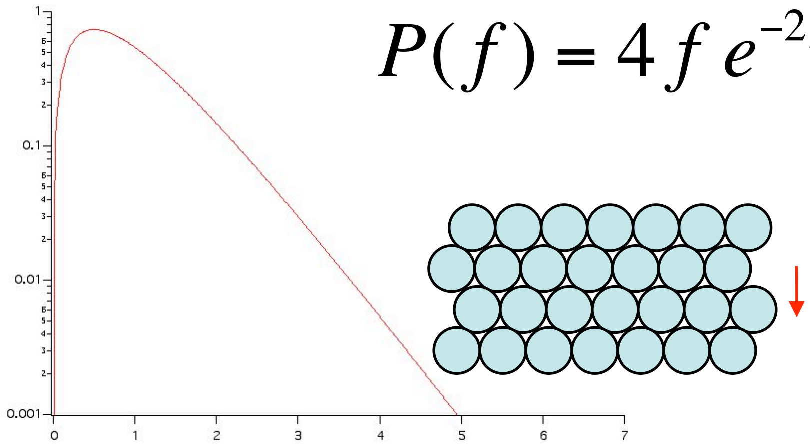


$$P(f) = 4 f e^{-2f}$$

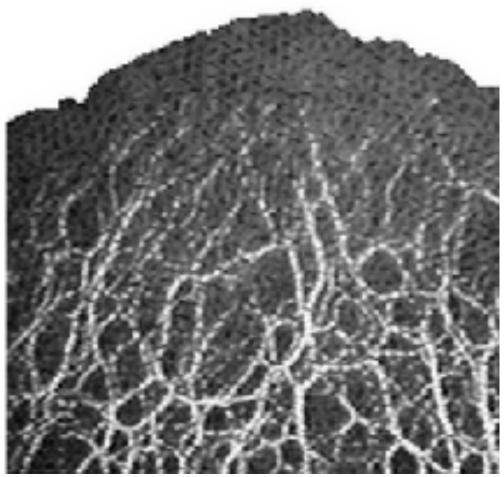
Liu et al, Science 1995
Coppersmith et al, PRE 1996

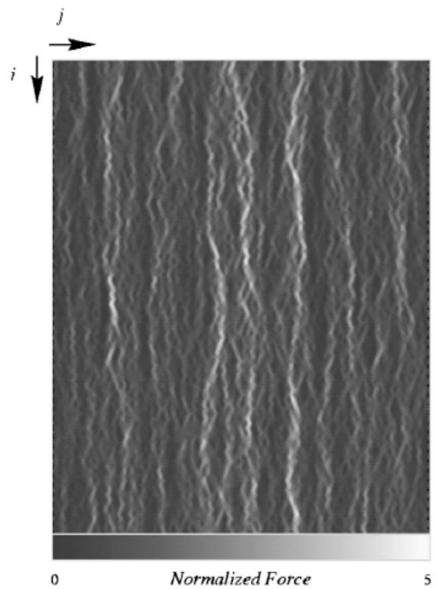
q-model



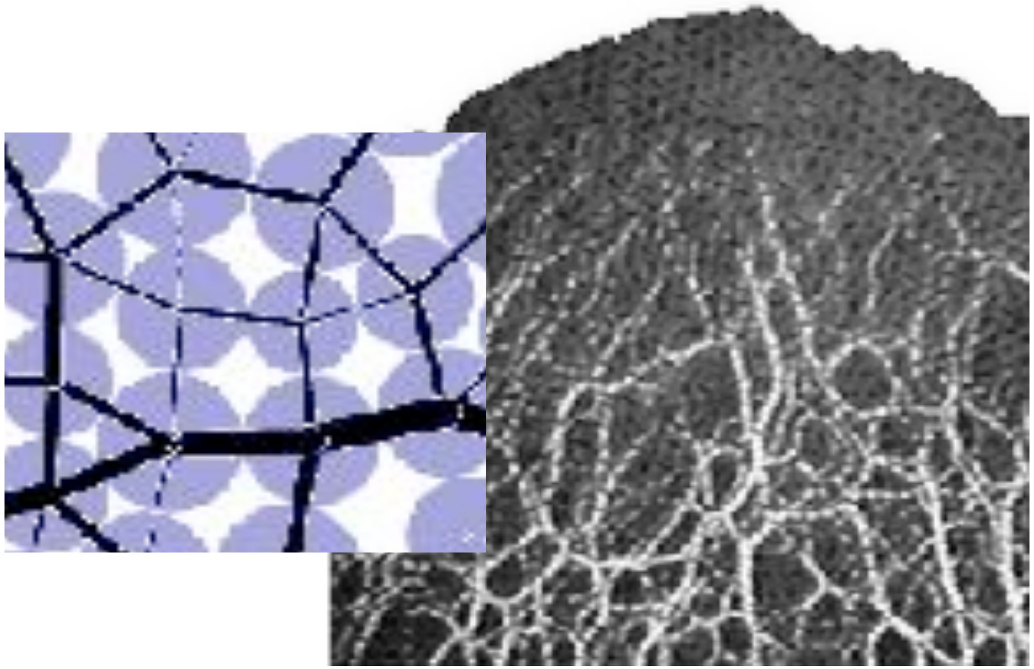


$$P(f) = 4\,f\,e^{-2f}$$





force balance: vector

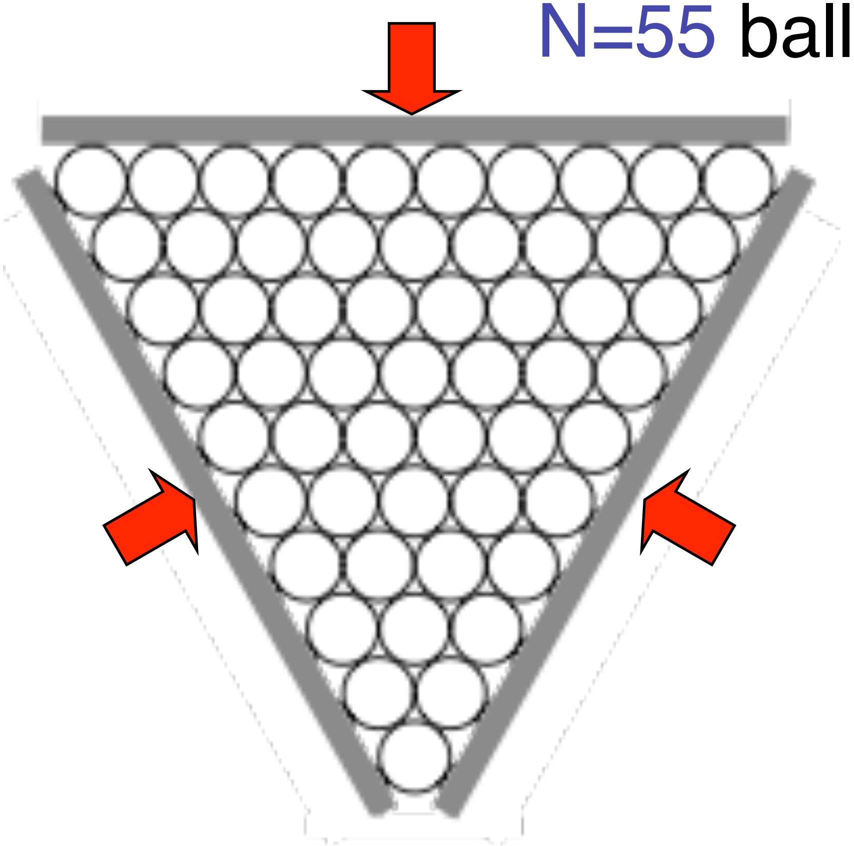


can we make a model based on vector balance?

snooker!

‘snooker’ packing

N=55 balls – frictionless forces

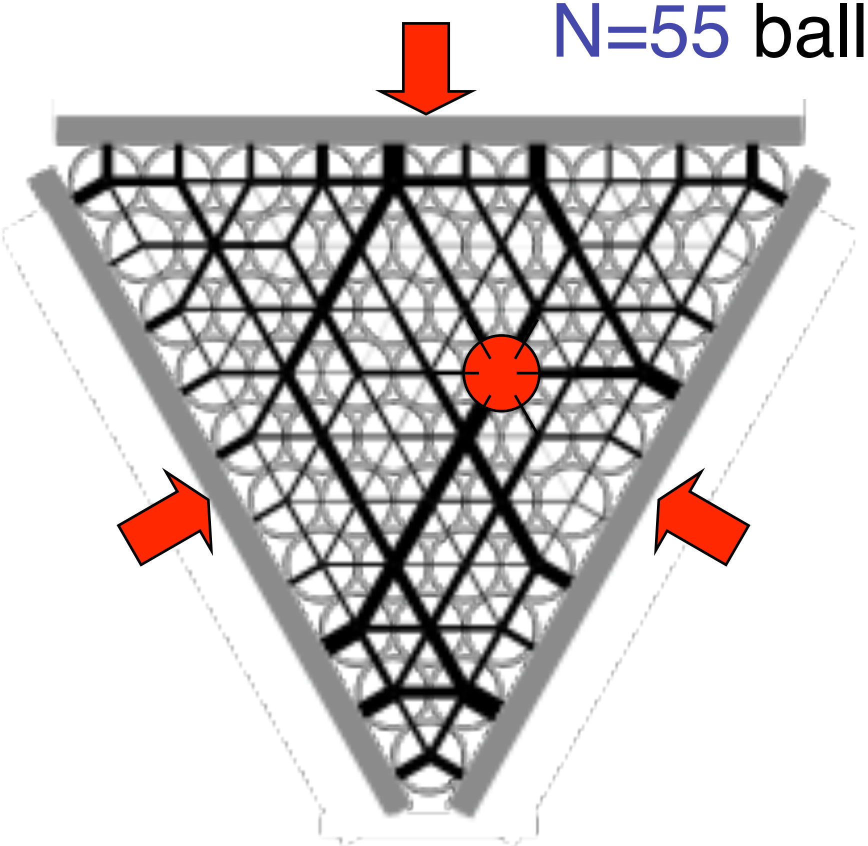


1. how many forces?

2. how many force balance equations? (balance x & y)

‘snooker’ packing

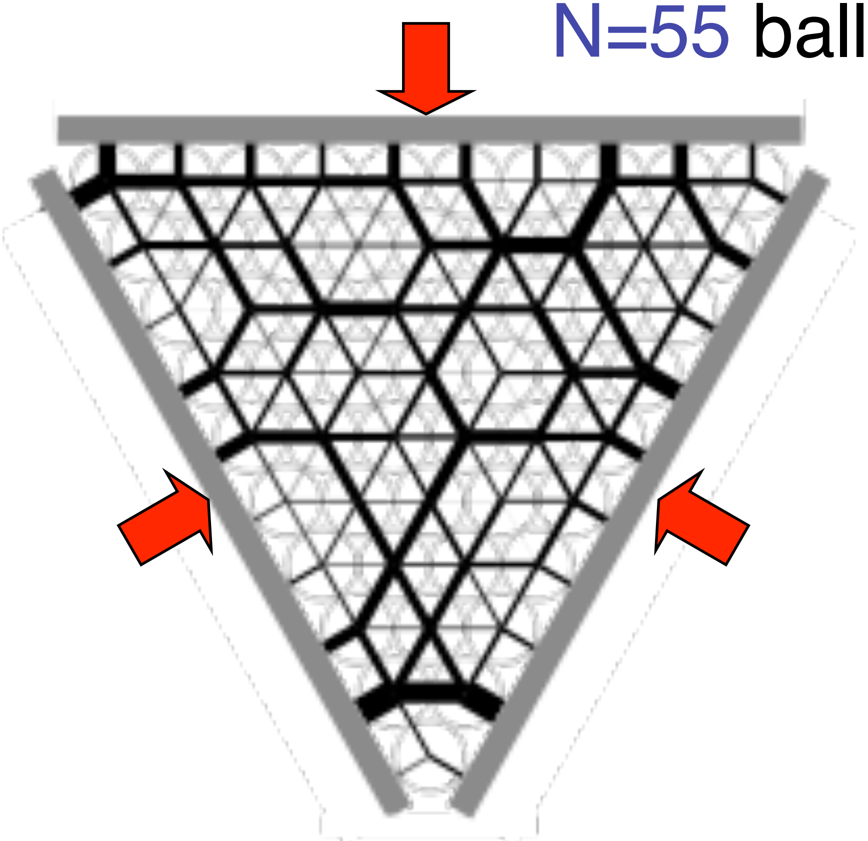
N=55 balls – repulsive forces



3N unknowns (forces)
2N eqs (balance x & y)

‘snooker’ packing

N=55 balls – repulsive forces

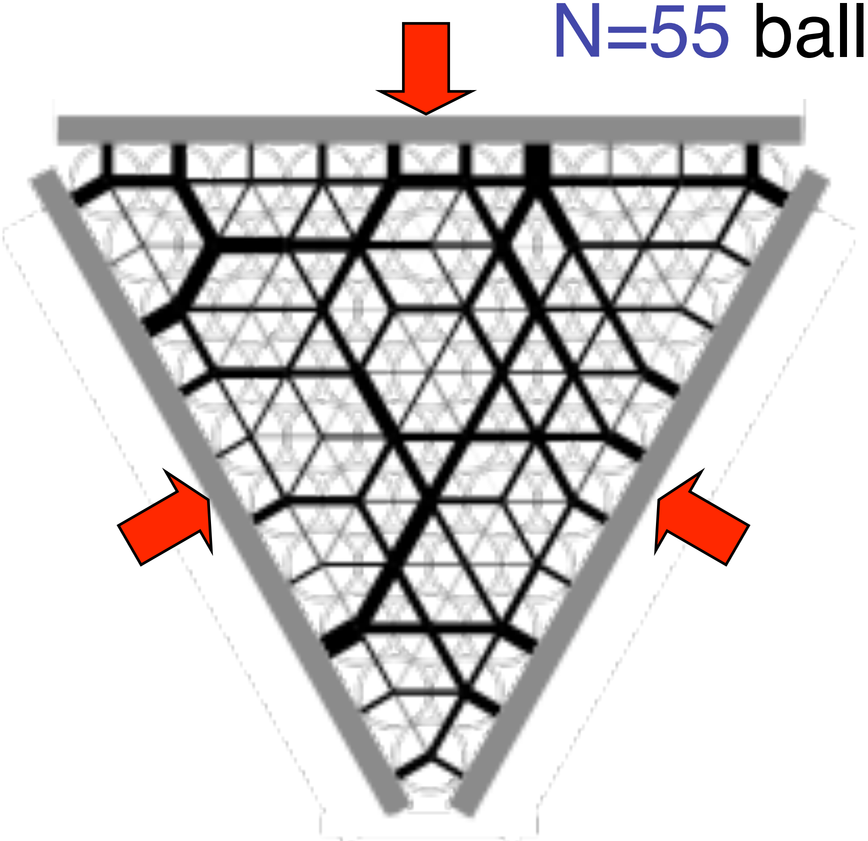


3N unknowns (forces)

2N eqs (balance x & y)

‘snooker’ packing

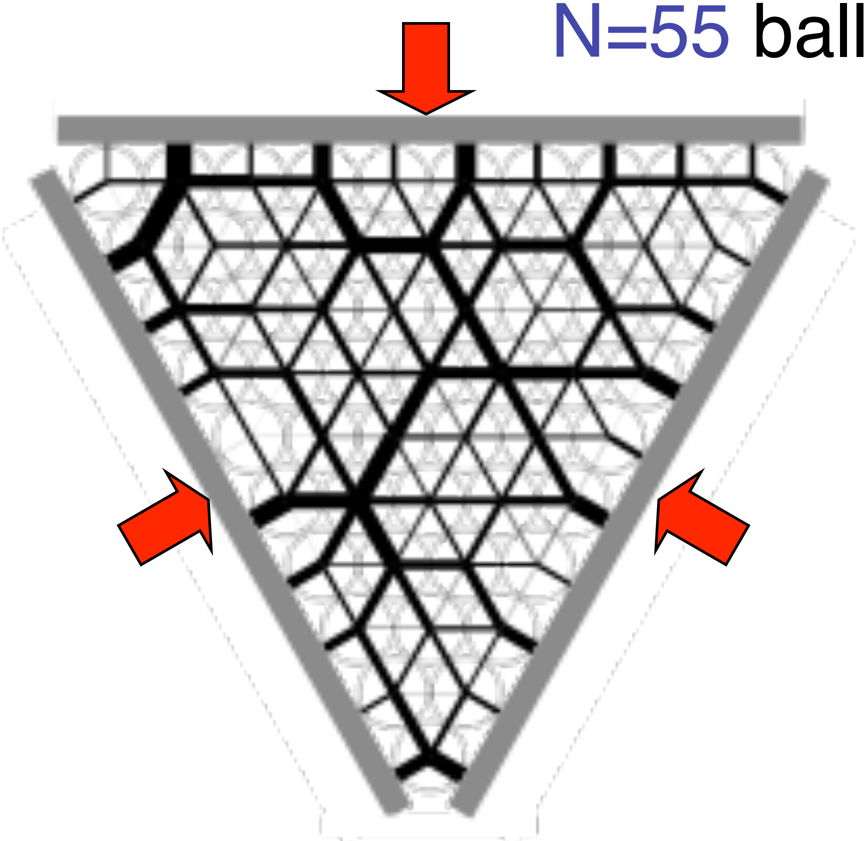
N=55 balls – repulsive forces



3N unknowns (forces)

2N eqs (balance x & y)

‘snooker’ packing

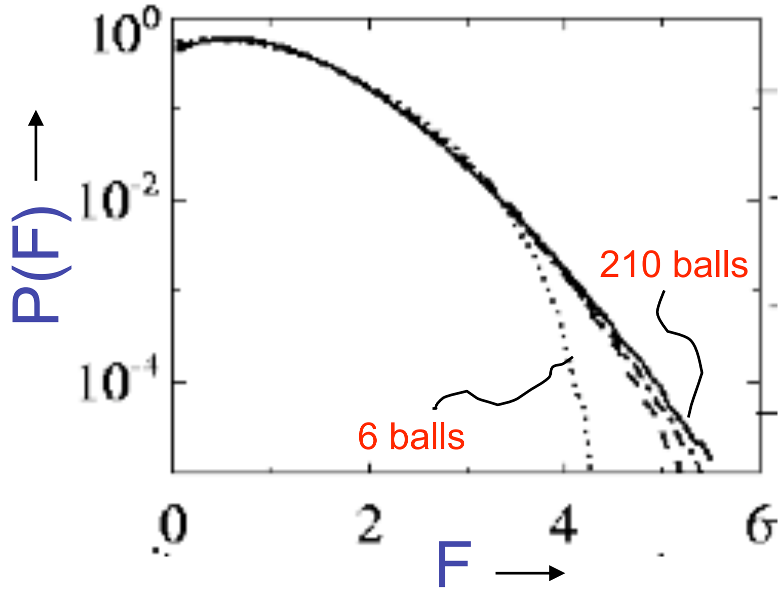


N=55 balls – repulsive forces

3N unknowns (forces)
2N eqs (balance x & y)

force statistics: P(f) ?

force statistics: P(f) ?

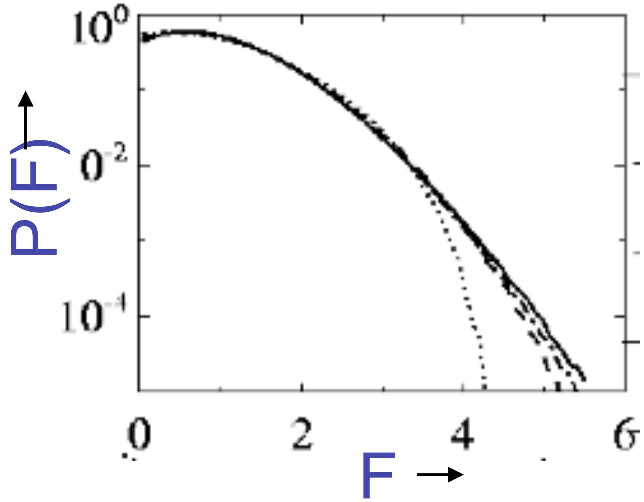


Snoeijer, Vlugt, van Hecke, van Saarloos, Phys. Rev. Lett. 2004
van Eerd, Ellenbroek, van Hecke, Snoeijer, Vlugt, Phys. Rev. E 2007

Tighe, Socolar, Schaeffer, Mitchener, Huber, Phys. Rev. E 2005
Tighe, van Eerd, Vlugt, Phys. Rev. Lett. 2008

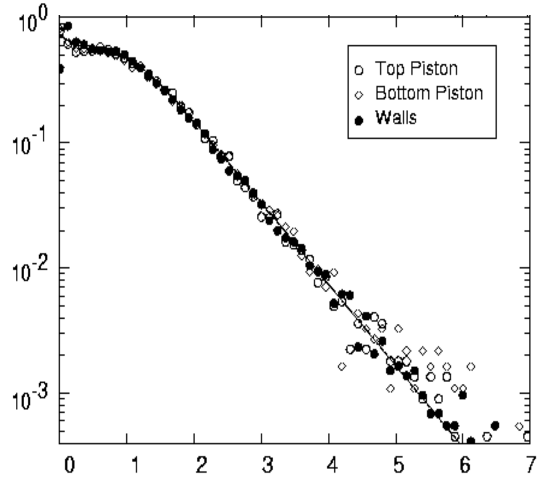
force statistics: P(f) ?

Theory: snooker



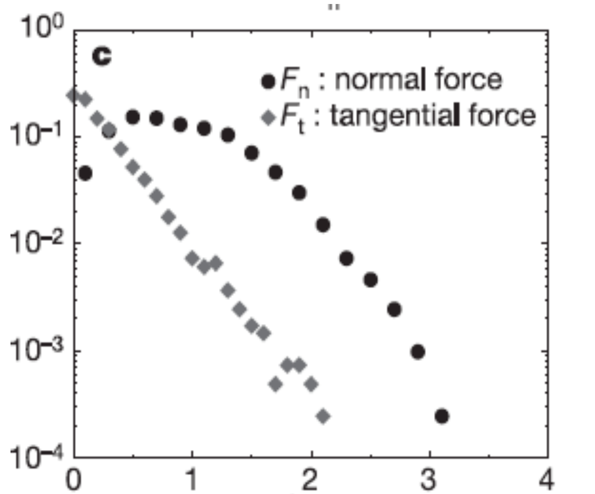
Experiments

glass beads (carbon paper)



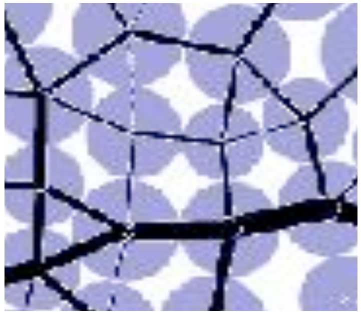
Liu *et al.*, Science 1995
D. Blair *et al.*, PRE 2001

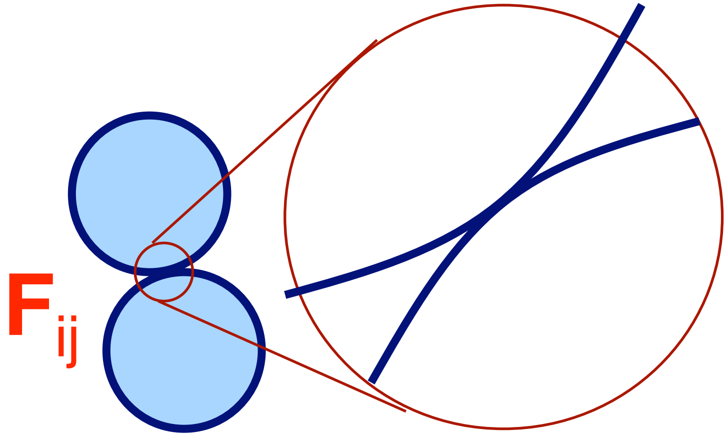
photo-elastic grains



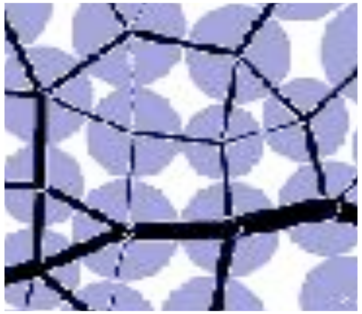
Majmudar & Behringer, Nature 2005

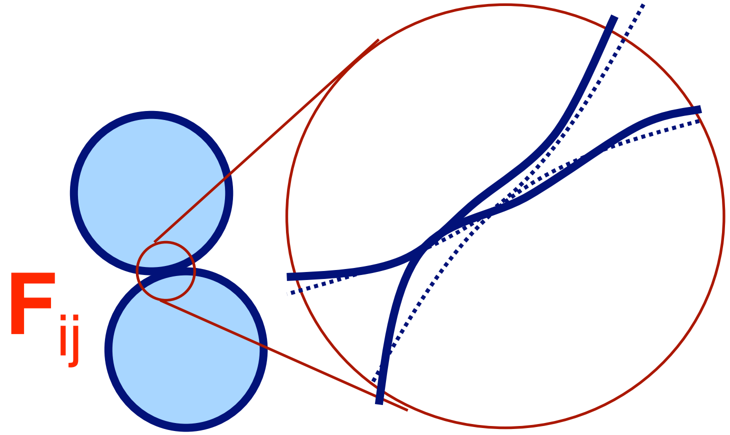
stochastic origin



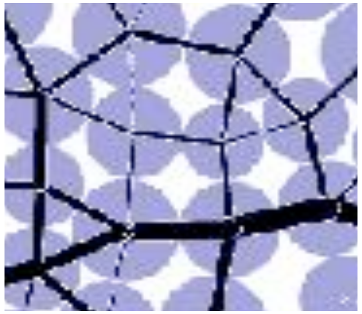


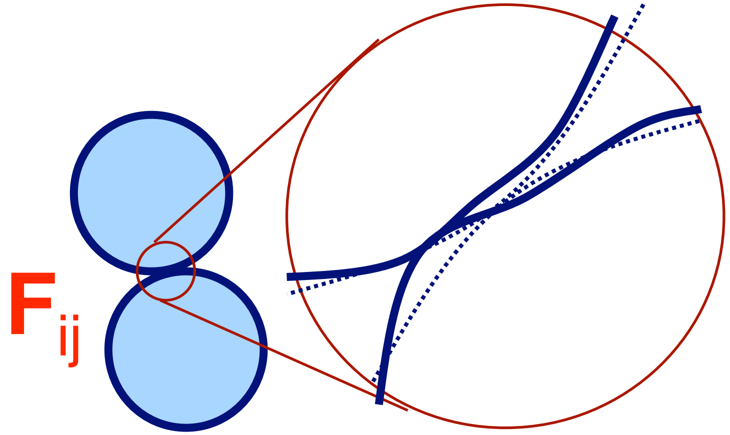
stochastic origin





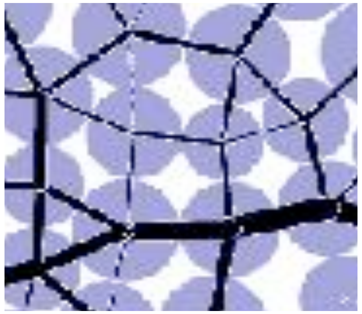
coordination number: z

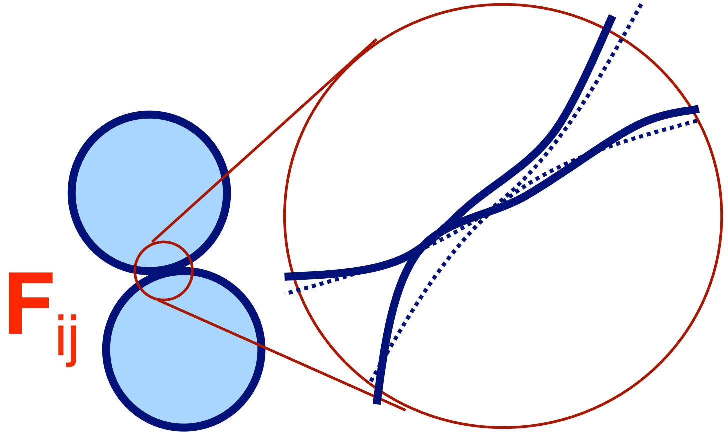




treat $\mathbf{F}_{ij}$ as ’unknown variable’: $zN/2$ unknowns

coordination number: z



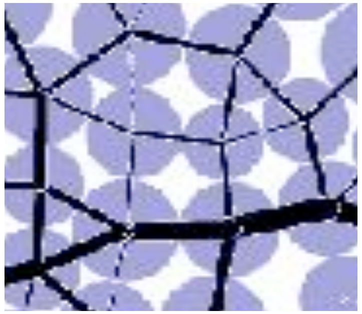


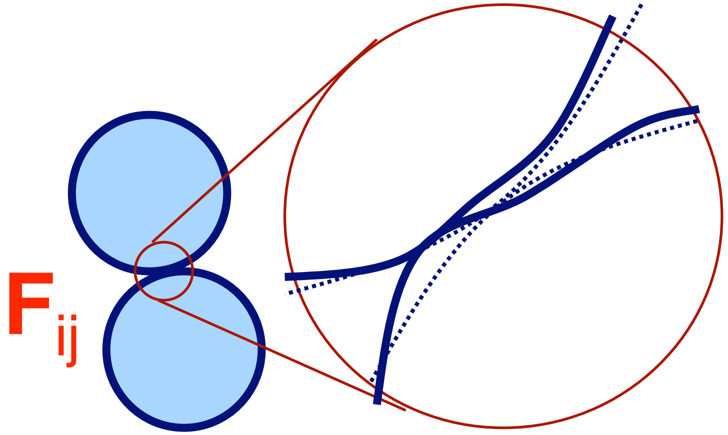
treat F_{ij} as 'unknown variable': $zN/2$ unknowns

mechanical equilibrium: $2N$ equations

(2 dimensions, frictionless particles)

coordination number: z





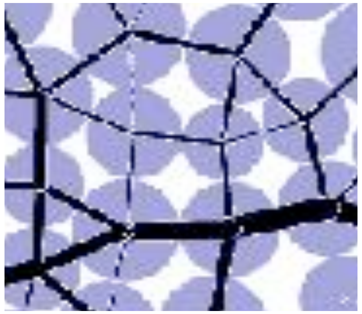
treat $\mathbf{F}_{ij}$ as 'unknown variable': zN/2 unknowns

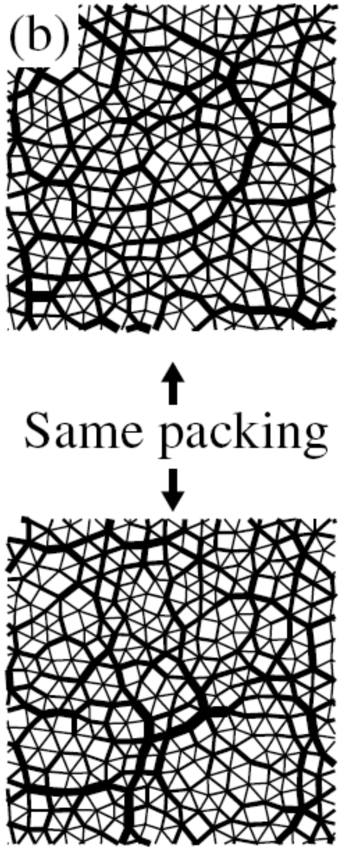
mechanical equilibrium: 2N equations

solutions exist if #unknowns ≥ #equations

$$z \geq 4$$

coordination number: z





z = 4: isostatic (unique force solution)

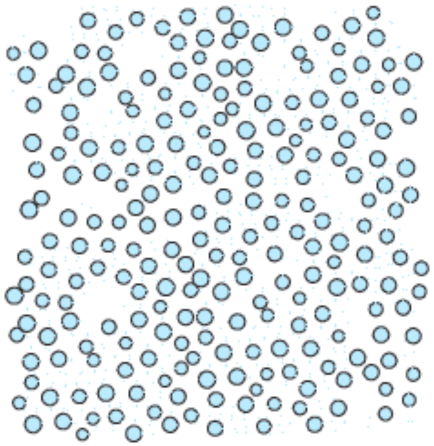
z > 4: hyperstatic
(many possible force solutions)

z < 4: no equilibrium possible

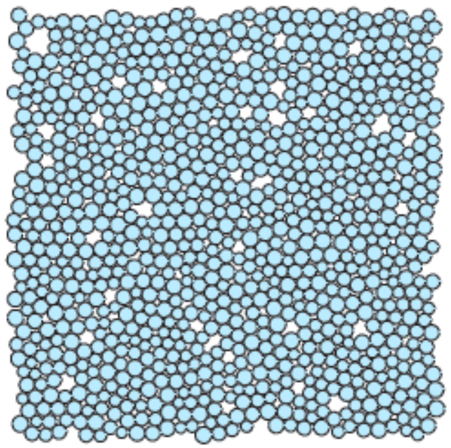
z = 5.5

frictionless packings

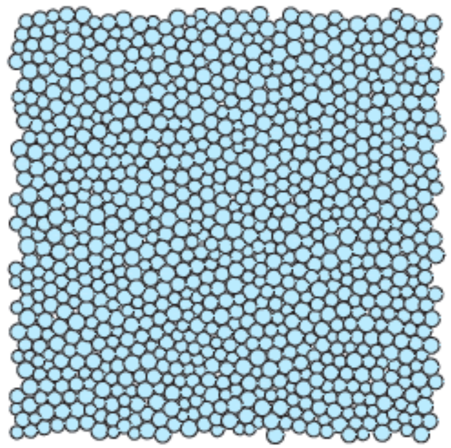
increasing density



loose grains



jamming point
'just touching'
$z = 4$!!

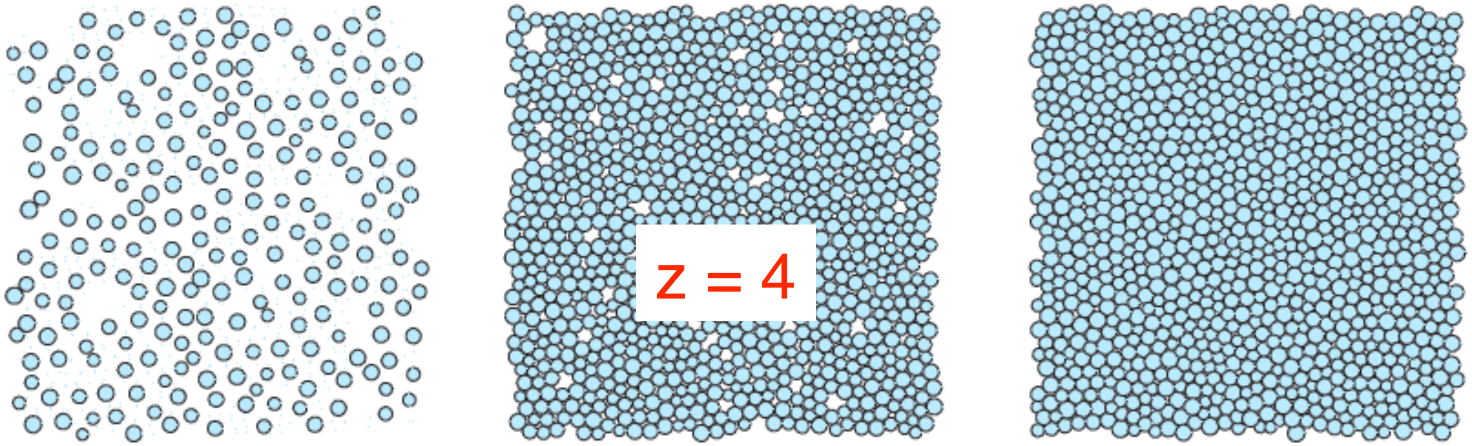


compressed

$z > 4$

frictionless packings

increasing density



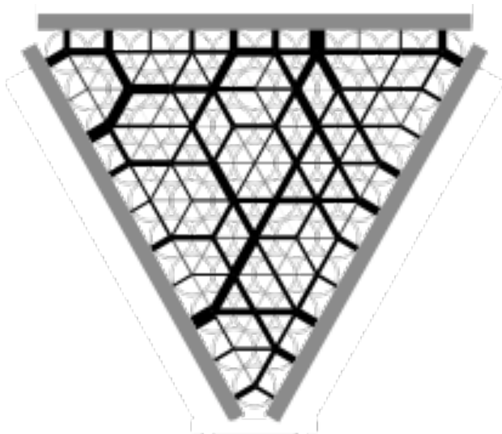
simple counting: nontrivial prediction for disordered system!

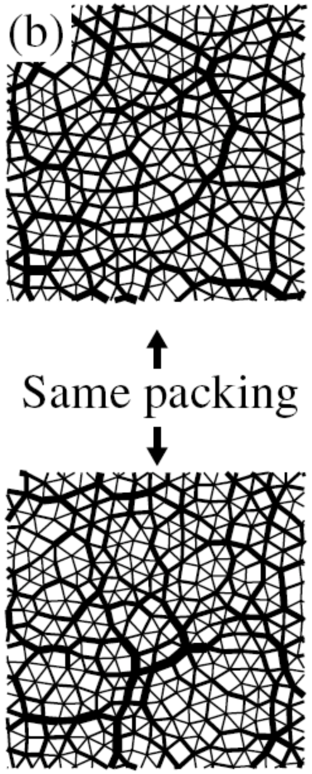
counting exercise

frictional contacts: exercise 3

what did we do?

- average over all possible realizations of force network, for given packing
- constraint: $\sum_{ij} f_{ij} = const$
- equal a priori probability



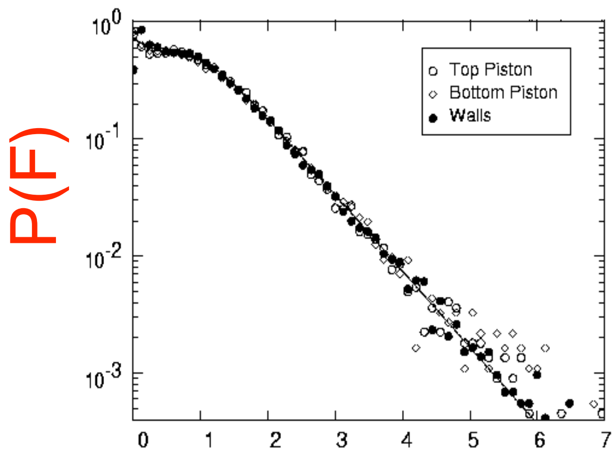


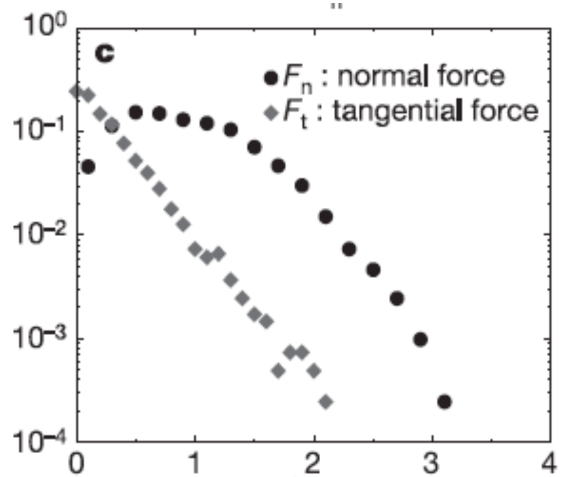
→ 'force network ensemble'

exponential or not?

carbon paper

photo-elastic grains

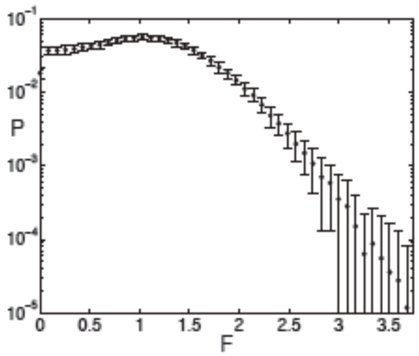




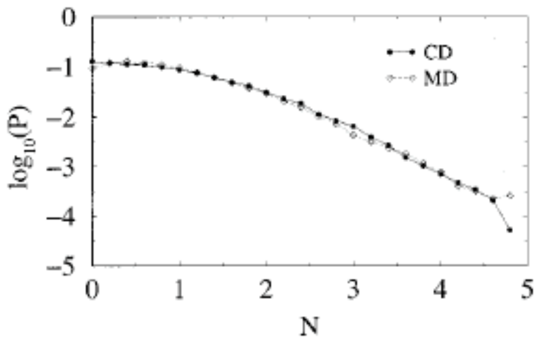
second opinion from numerical simulations

molecular dynamics...

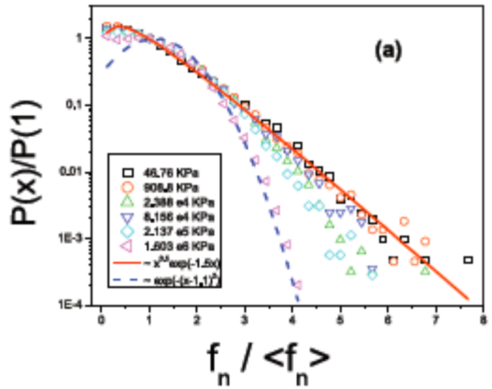
exponential or not?



Goldenberg and Goldhirsch,
Gran. Matt. 2004

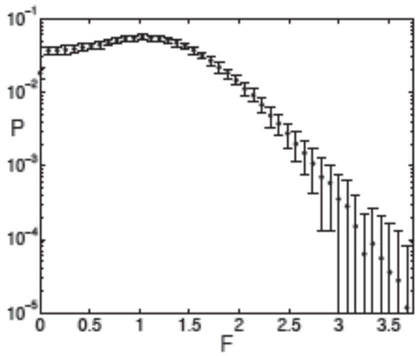


Radjai et al., Chaos 1999

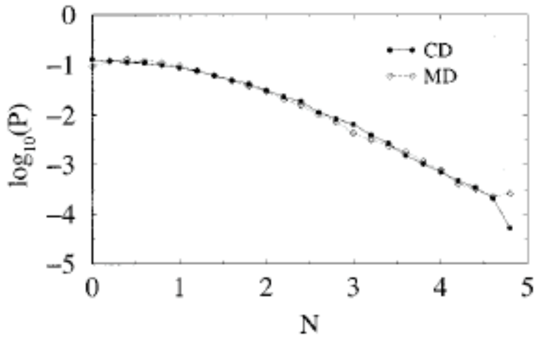


Zhang and Makse, PRE 2004

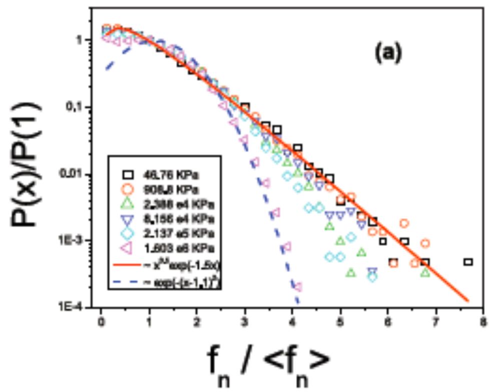
exponential or not?



Goldenberg and Goldhirsch,
Gran. Matt. 2004



Radjai et al., Chaos 1999



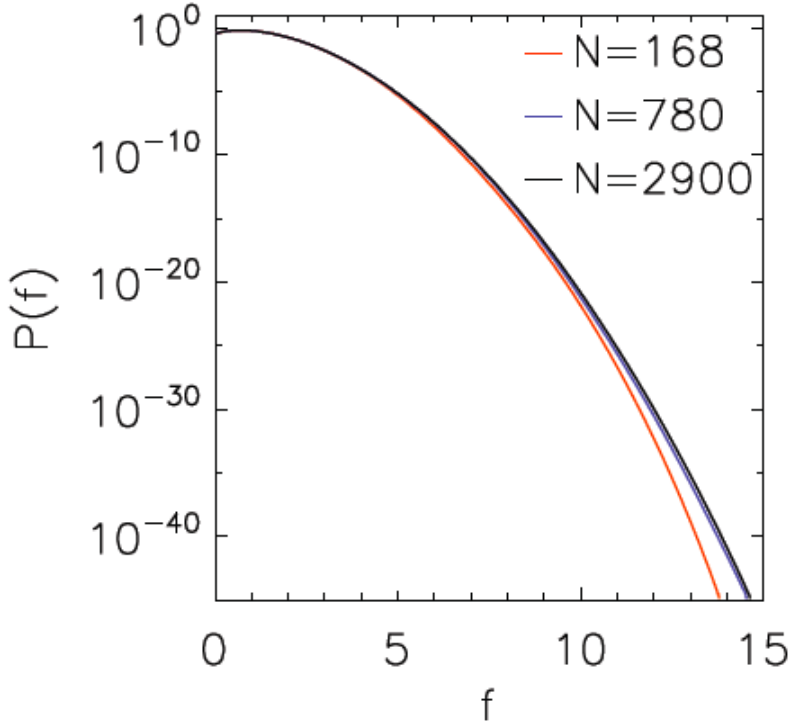
Zhang and Makse, PRE 2004

third opinion: force network ensemble

tail of P(F)

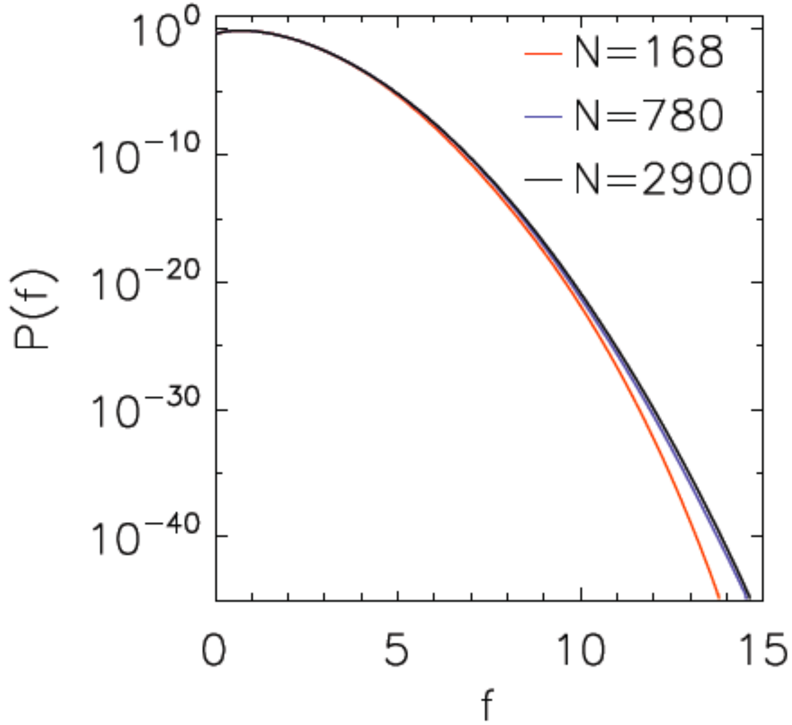
ensemble: faster than exponential

(for 2D, 3D, ordered, disordered, frictional, frictionless)



tail of P(F)

ensemble: faster than exponential

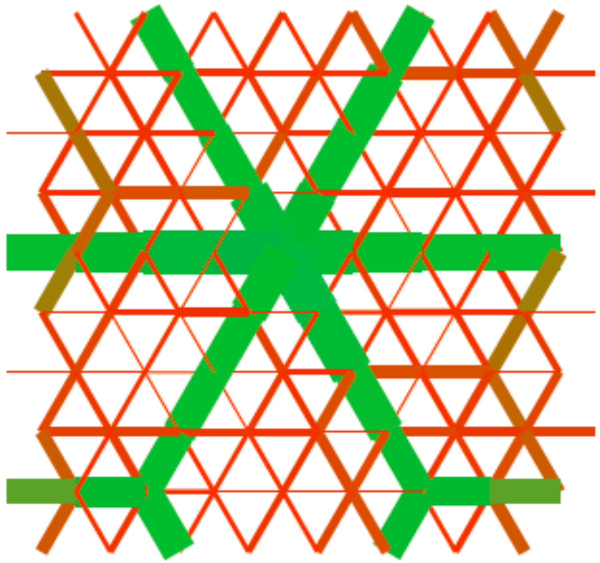


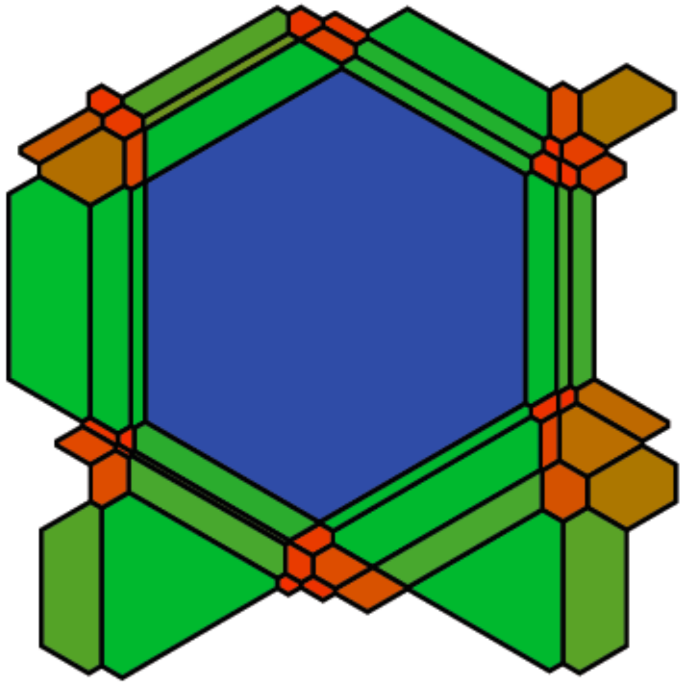
what is wrong with
'conservation of force/Boltzmann' argument?

...

slides by brian tighe

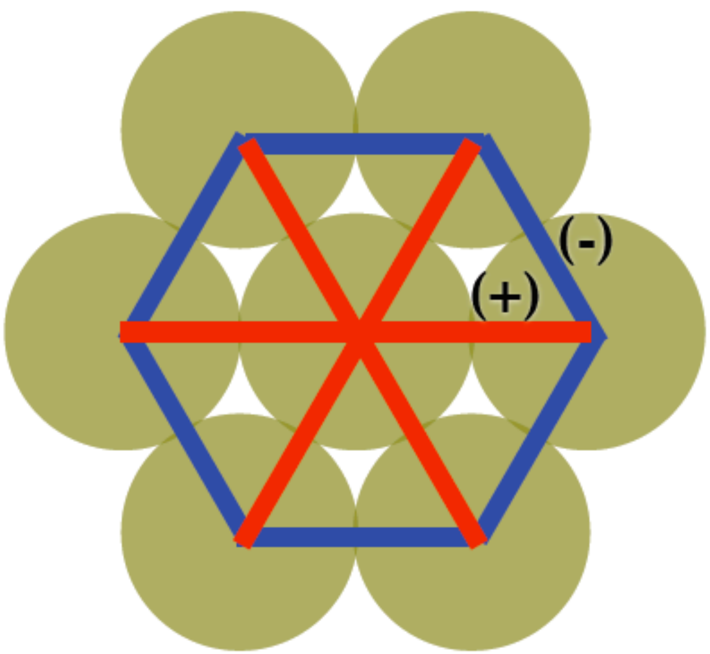
reciprocal tiling





J. C. Maxwell
Phil. Mag. 1864

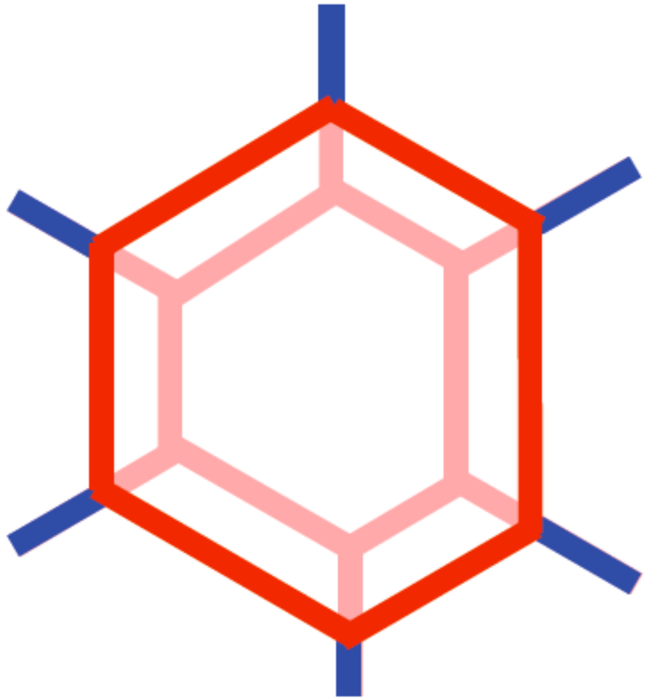
reciprocal tiling



BPT, Socolar, et al.
PRE 2005

real space

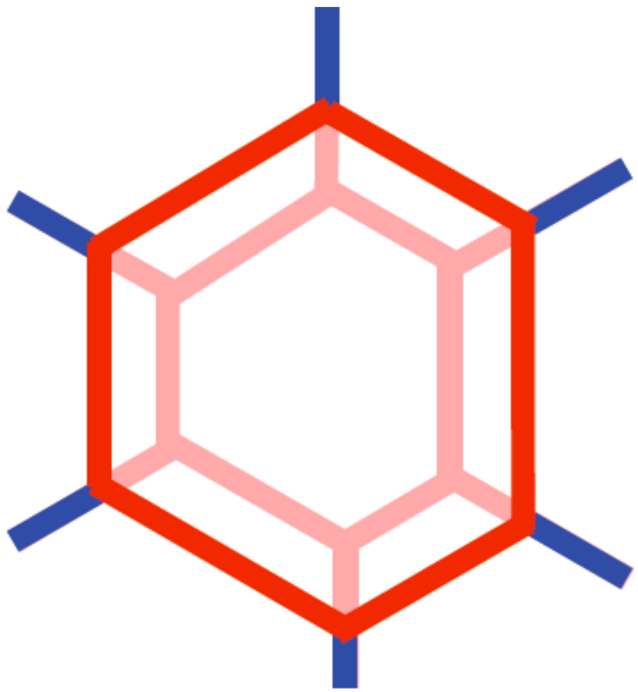
redistribute forces



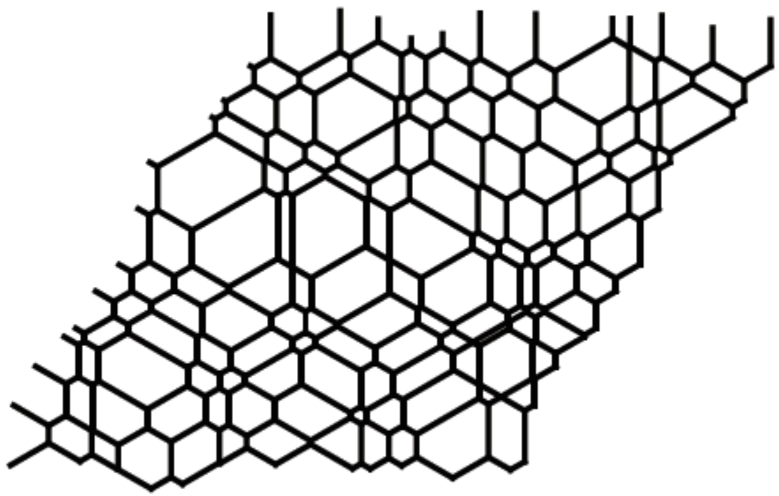
reciprocal tiling

redistribute areas

global constraints



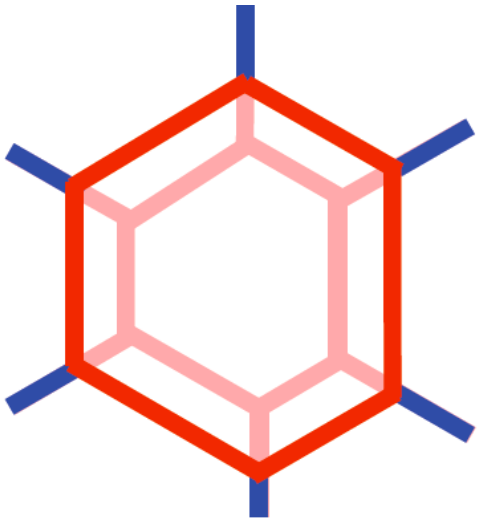
redistribute areas



pressure conserved

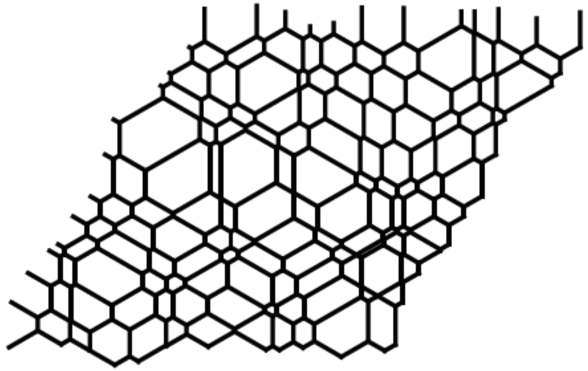
area conserved

global constraints



redistribute areas

$$p_i = \sum_j f_{ij}$$



pressure conserved

area conserved

$$\sum_i p_i = const$$

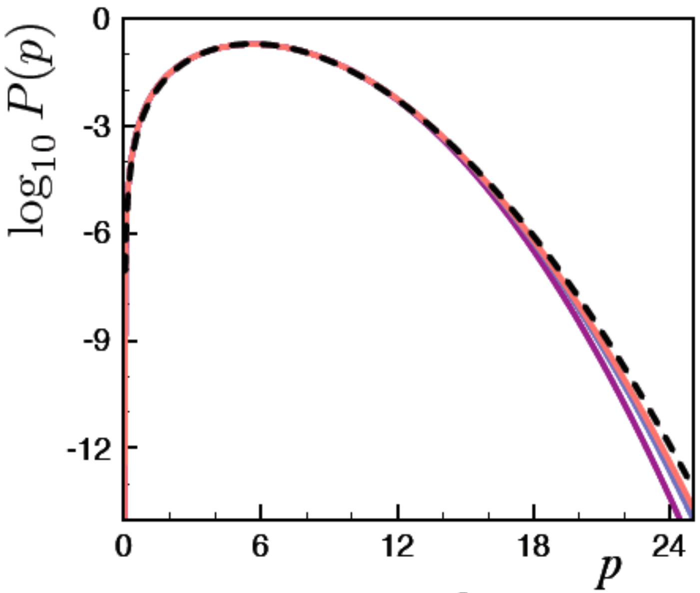
$$\sum_i p_i^2 = const$$

entropy maximization

$$P(p) \sim p^3 \exp(-ap - bp^2)$$

theory
vs.
numerics

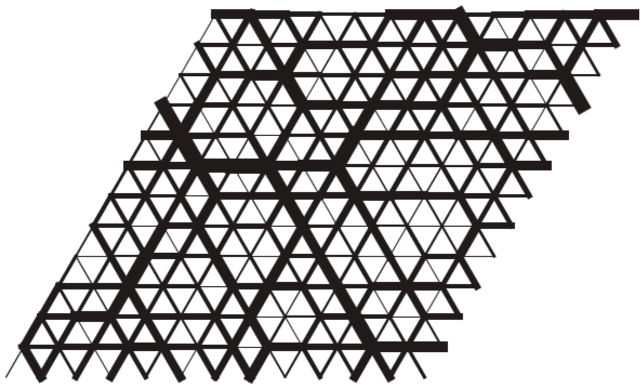
BPT, van Eerd, and Vlugt
PRL 2008

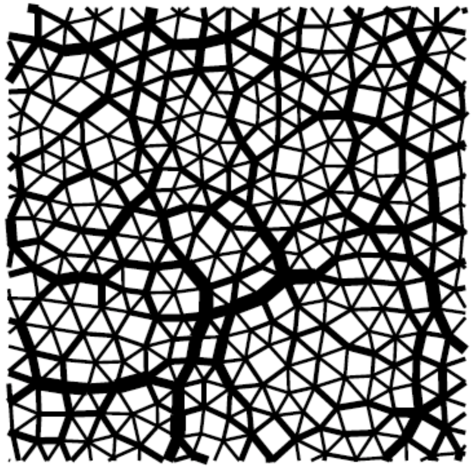


conclusion

- granular solids: intricate distribution of contact forces
- force network ensemble: statistical tool

- unresolved questions:
    - stress propagation (stress dip)
    - isostatic limit (Brian Tighe)





shear in ensemble

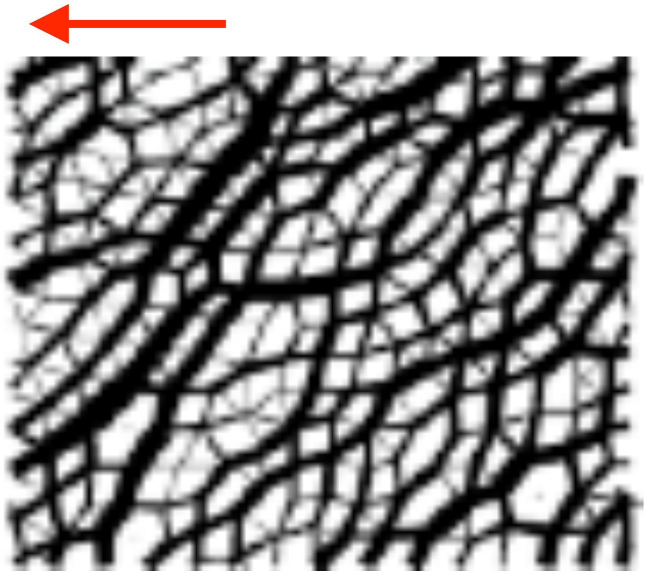
ensemble

shear stress τ♪

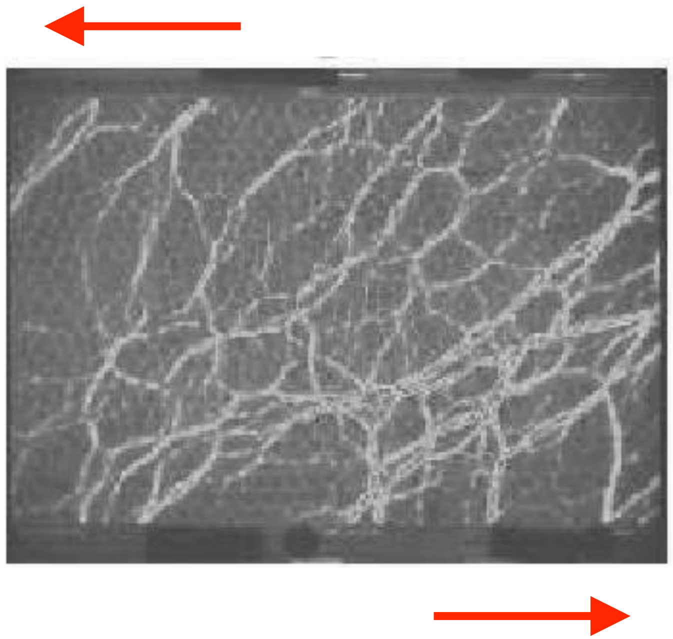
force anisotropy

shear in ensemble

ensemble



experiment



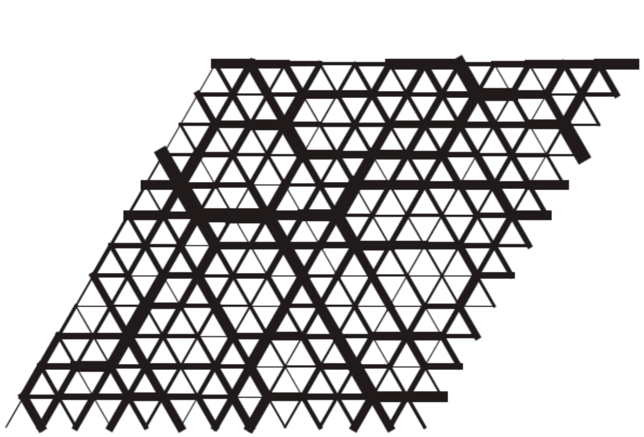
shear stress τ♪

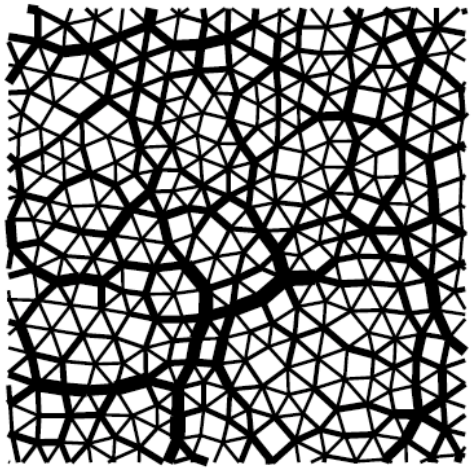
J. Geng *et al.,* Physica D ('03)
Atman *et al.,* Eur. Phys. J. E ('05)

force anisotropy

ensemble ideas

- great statistical tool to study force networks,
  ordered, disordered
- generalize to ensemble of packings (Edwards)





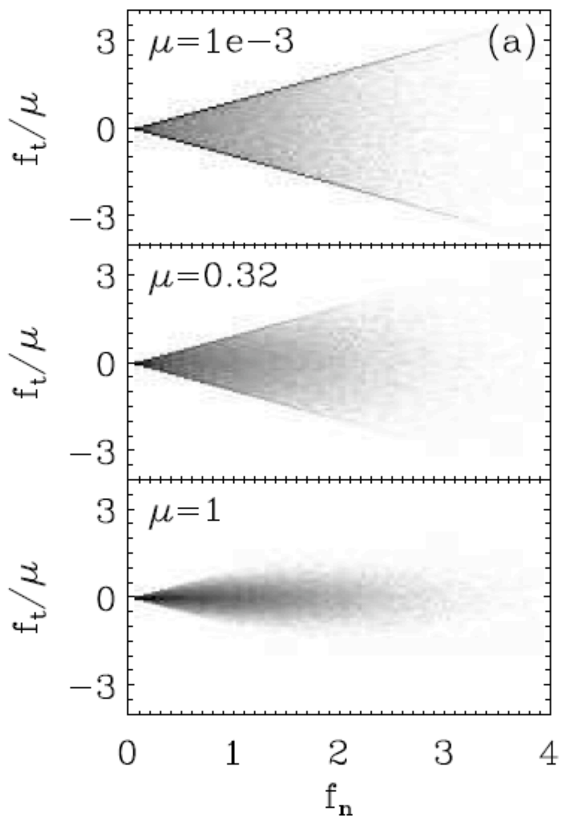
- does not describe isostatic regime: see next lecture
- friction ?

friction: Coulomb cone

Coulomb friction:

$$\left| f_t \right| \leq \mu f_n$$

MD result



Shundyak et al PRE 2007

friction: Coulomb cone

Coulomb friction:

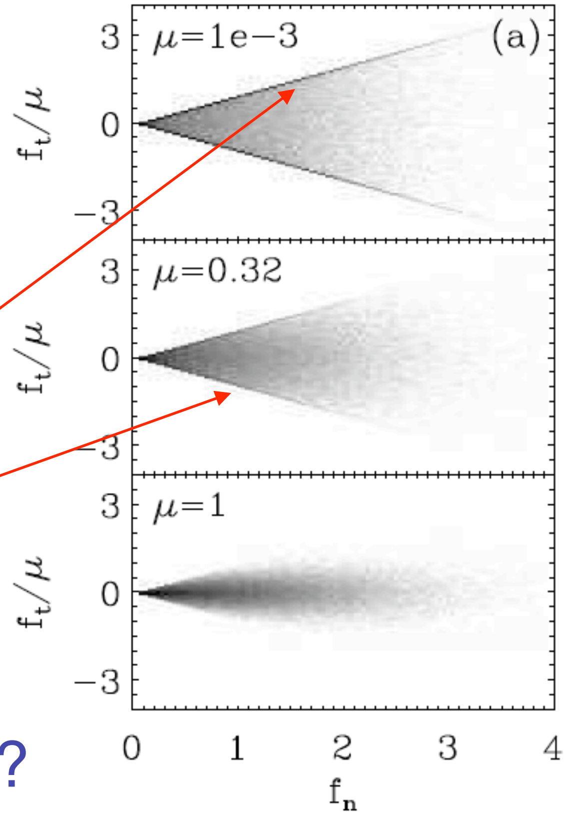
$$|f_t| \leq \mu f_n$$

finite fraction of forces that have

$$|f_t| = f_n$$

flat measure assumption ??

MD result



Shundyak et al PRE 2007