Jamming

JMBC course on Granular Matter
11 Feb 2010

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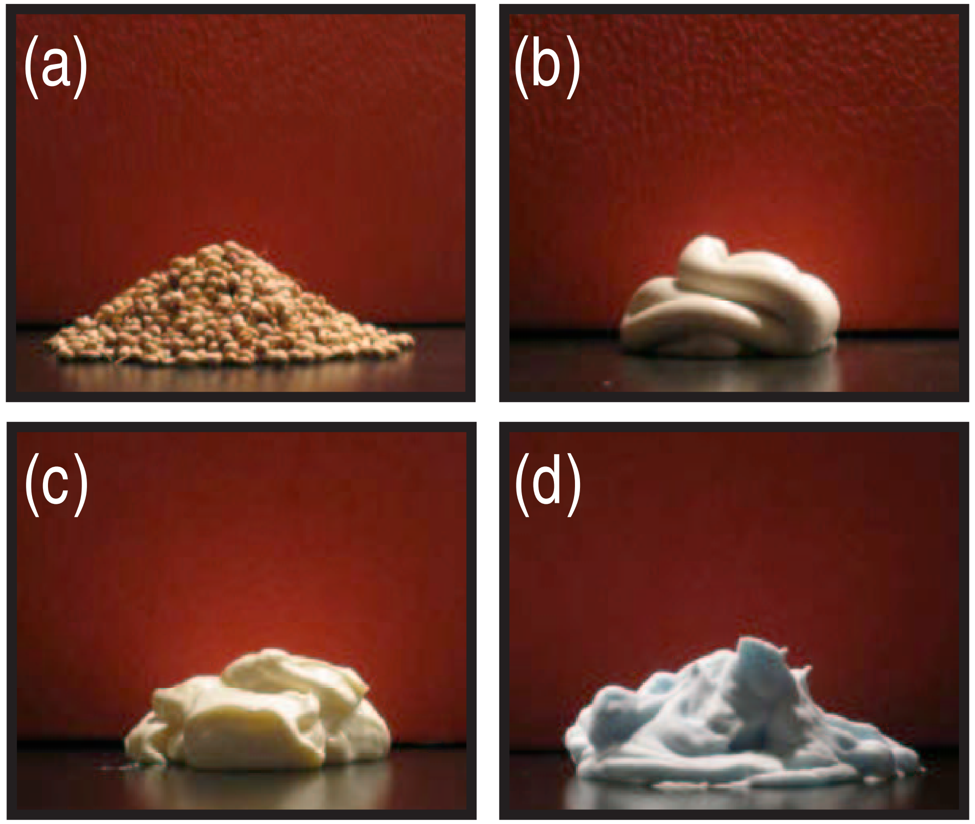
Or, what do sand, foam, emulsions, colloids, glasses etc. have in common?

Or, transitions to rigidity in disordered media



What we're talking about

disordered materials (un)**jam** when they lose rigidity

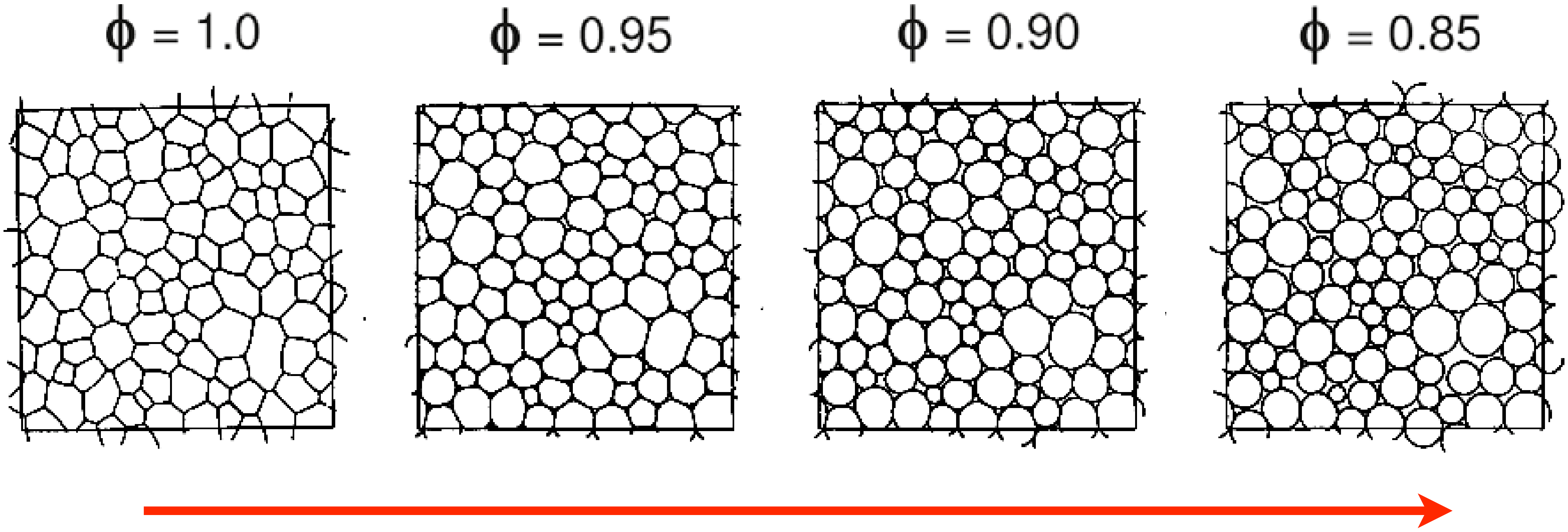
distance to jamming governs **geometric, mechanical, vibrational** and **rheological** properties

soft spheres - the Ising model of jamming

plus a **break** after 45 min.

Two Gedankenexperimente

1. Making foam “wet”



$$\phi = \frac{\text{area of bubbles}}{\text{area of cell}}$$

Bolton & Weaire,
PRL 1990

Two Gedankenexperimente

1. Making foam "wet"

Eventually the pressure goes to zero -- the material has "**unjammed**"

dry
large ϕ
high pressure

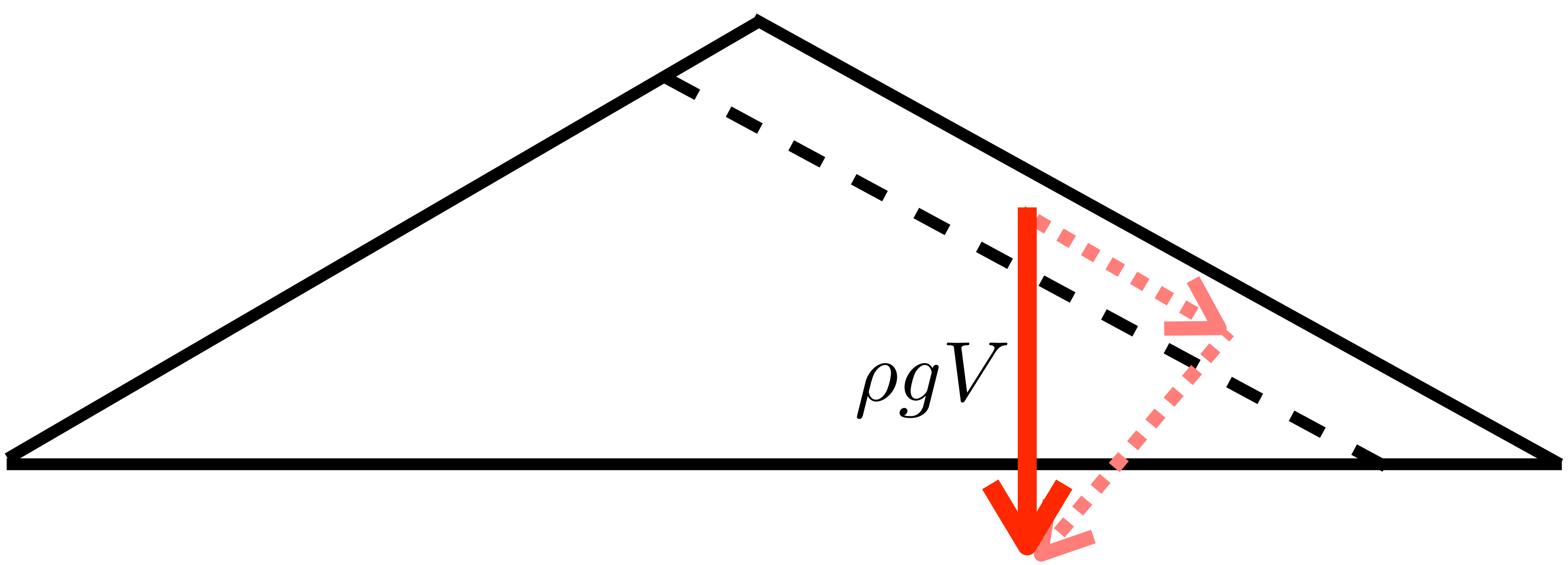
wet
small(er) ϕ
low pressure

Two Gedankenexperimente

2. Tilting a sandpile

surface is being sheared

= force parallel to surface



Two Gedankenexperimente

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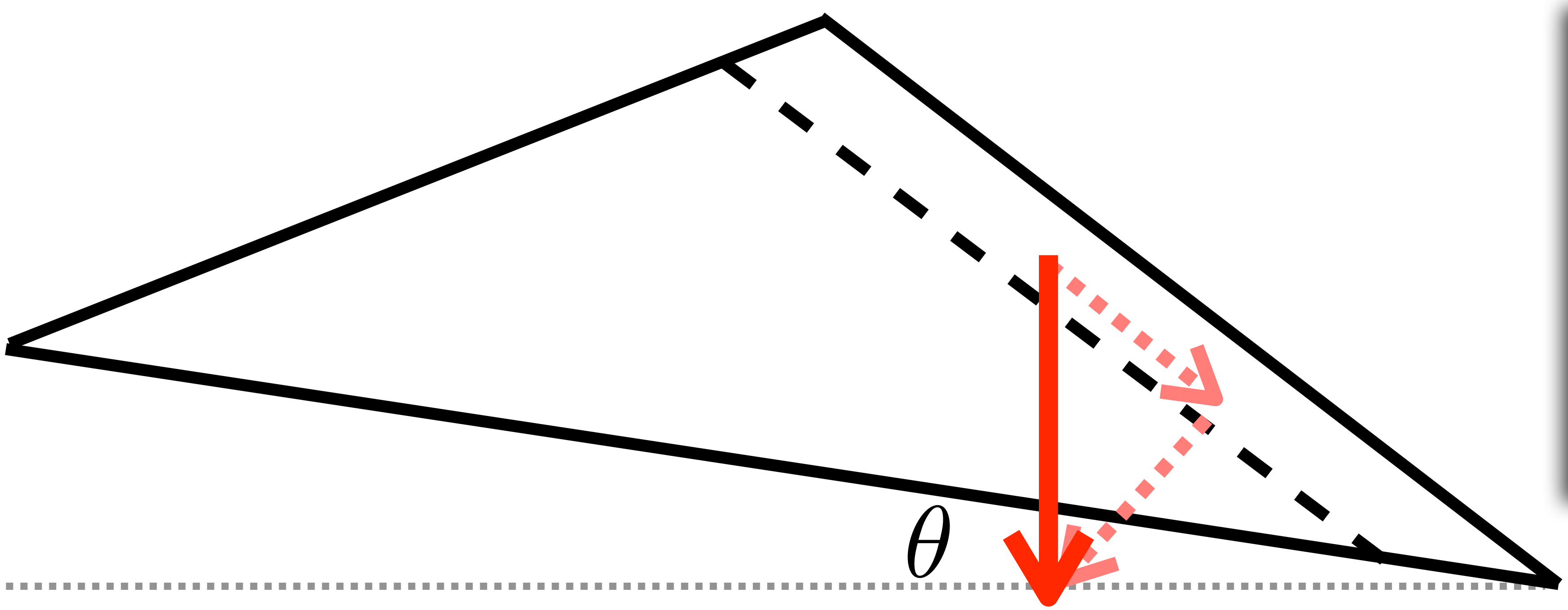
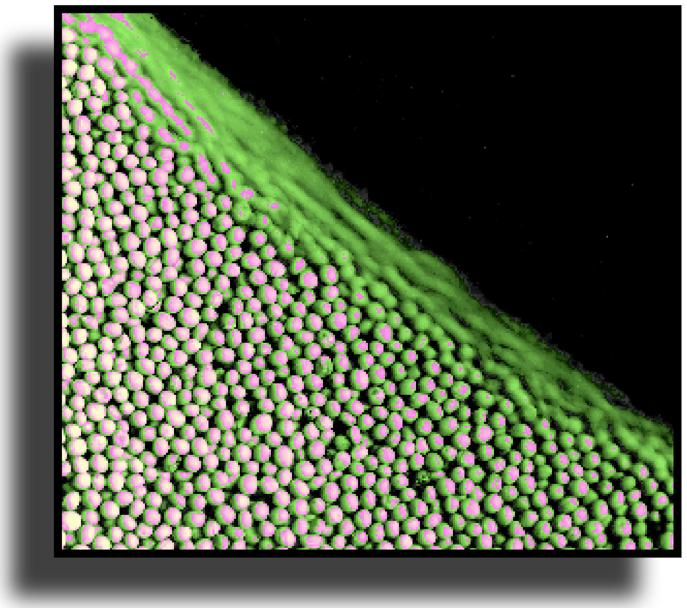
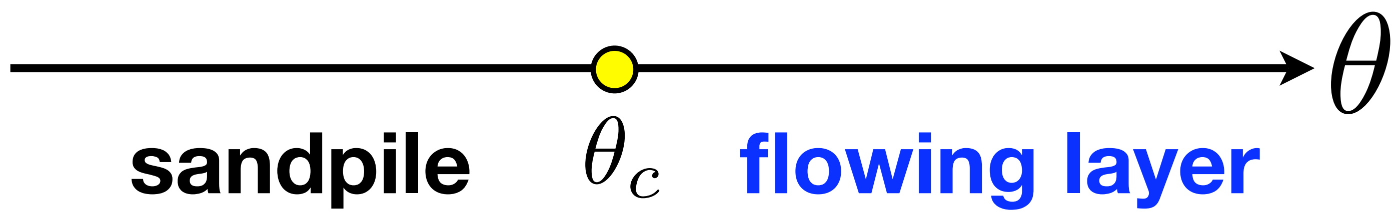
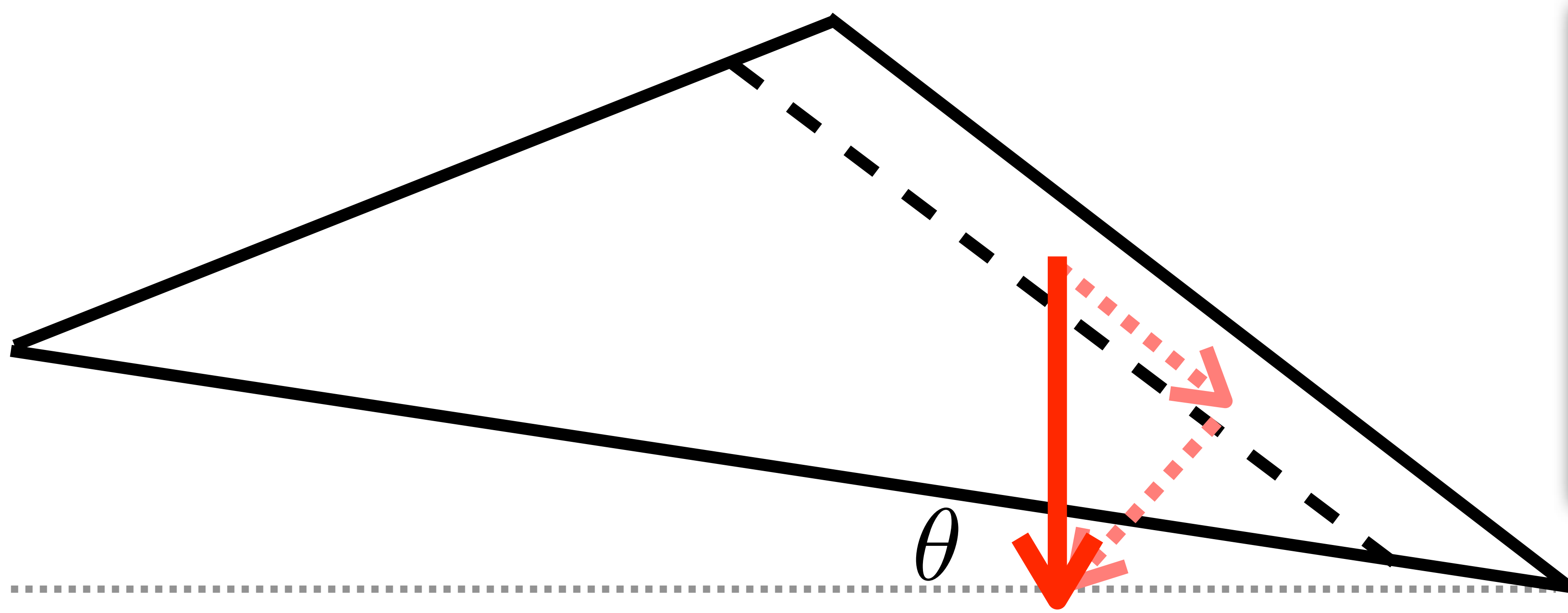


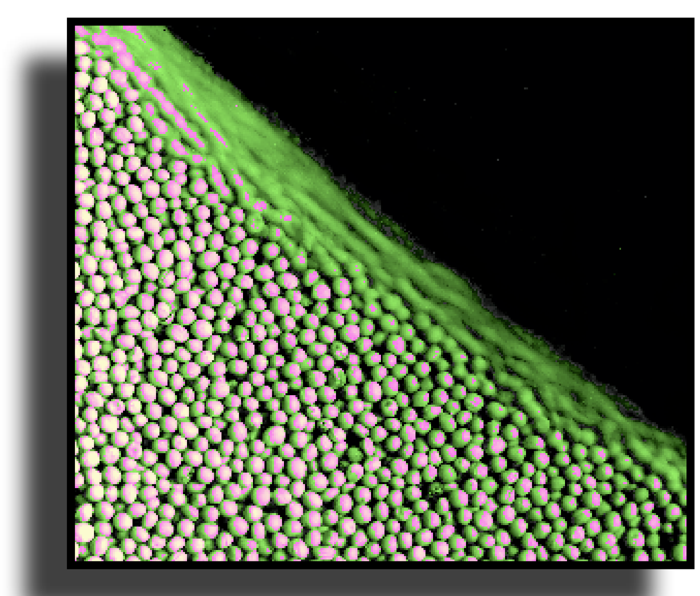


Two Gedankenexperimente

2. Tilting a sandpile







Flowing grains:
Jaeger and Nagel,
U. Chicago

Two Gedankenexperimente

2. Tilting a sandpile

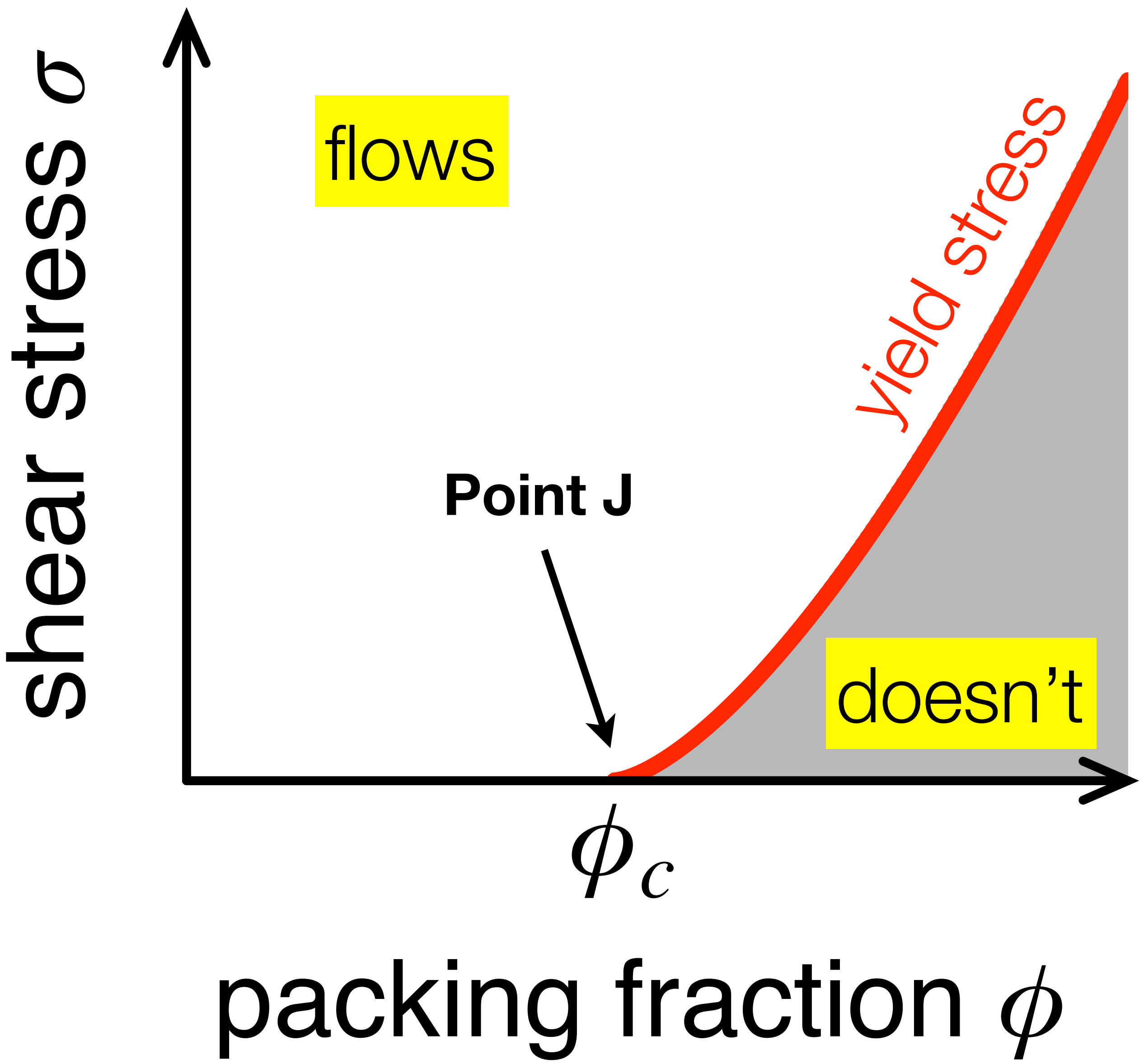
more generally:

$$\sigma_{\text{yield}} = \sigma(\theta_c) \qquad \sigma = \sigma(\theta)$$

sandpile **flowing layer**

$$\text{shear stress} = \frac{\text{shearing force}}{\text{surface area}}$$

Jamming and rigidity



transition between states

rigid ⇒ not rigid

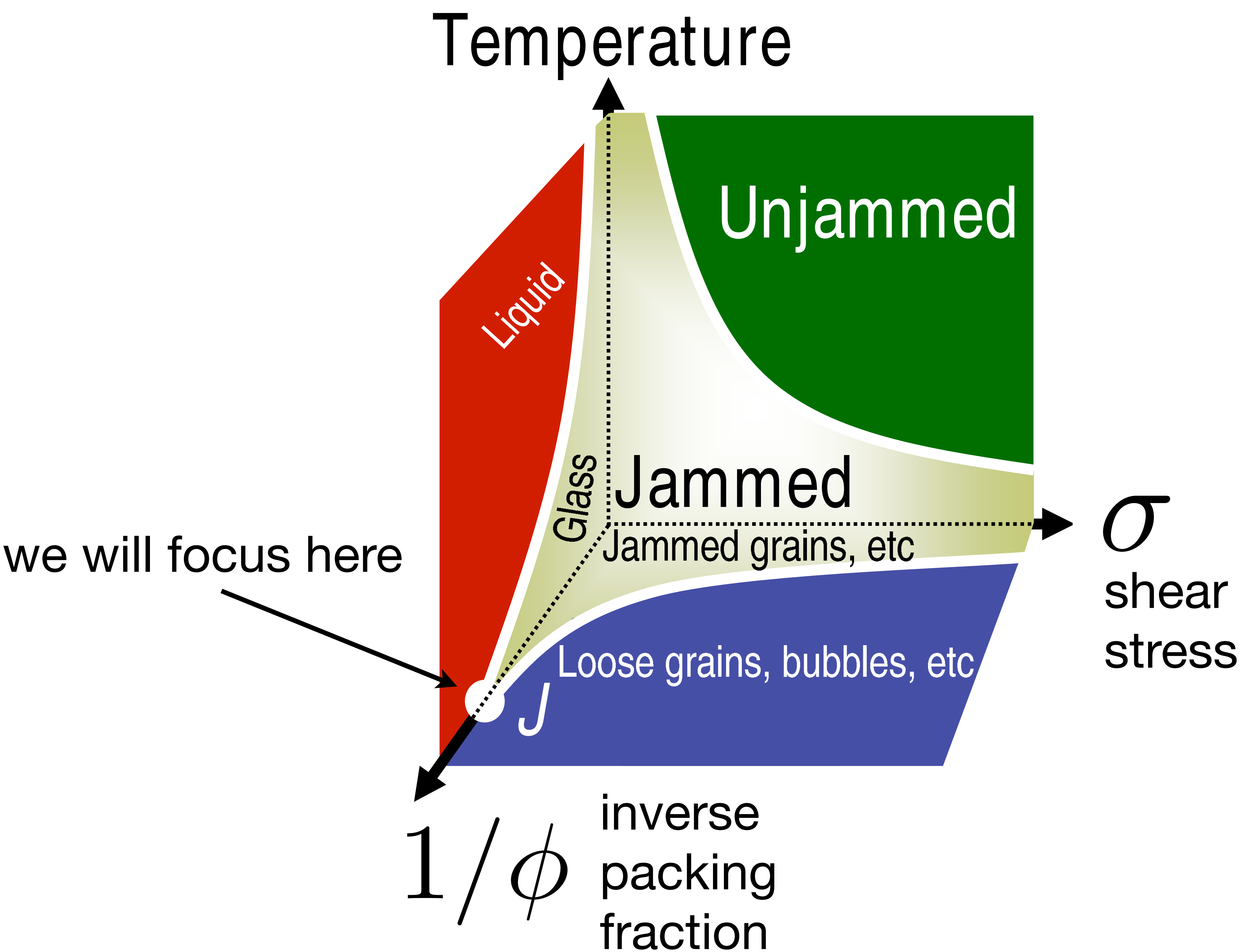
disordered ⇒ disordered

without changing temperature!

Jamming and rigidity

Liu & Nagel, Nature 1995

“new and improved” version
van Hecke 2010



Why study jamming?

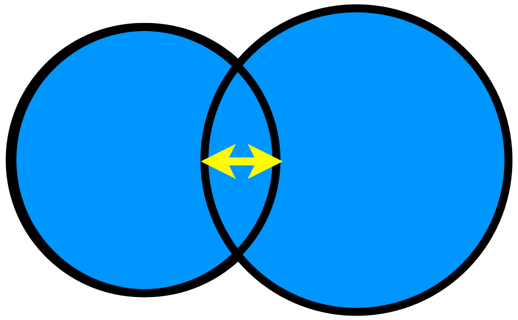
Nonequilibrium phase transition

⇒ (hope for) **universality**

many properties governed by one attribute: **distance to transition**

some materials (or models) more **convenient** than others

Soft spheres: Model system



assumptions

all particles are spheres

particles can deform
-- **not** perfectly rigid

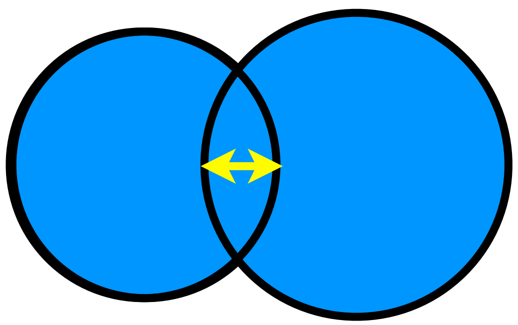
contact forces only

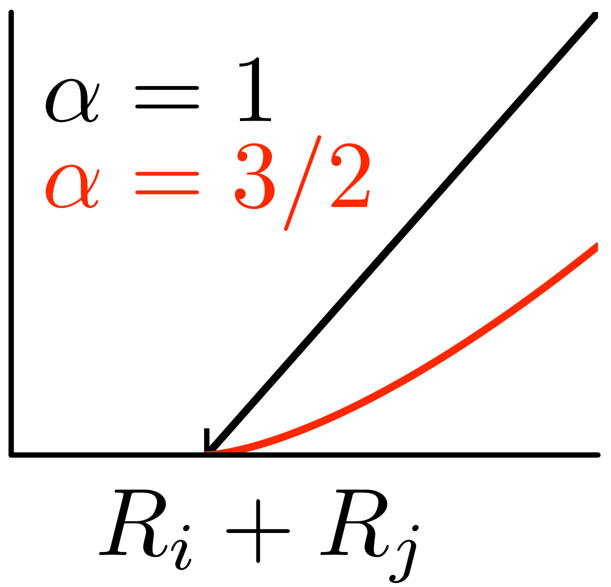
repulsive forces only

no friction!

Soft spheres: Model system

$$f_{ij} = k(R_i + R_j - |\vec{r}_i - \vec{r}_j|)^\alpha$$





"Hookean"

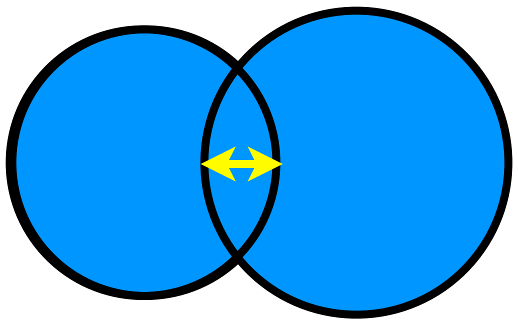
"Hertzian"

force a function of overlap of spheres

not touching: no force

Soft spheres: Local vs. Global

Local

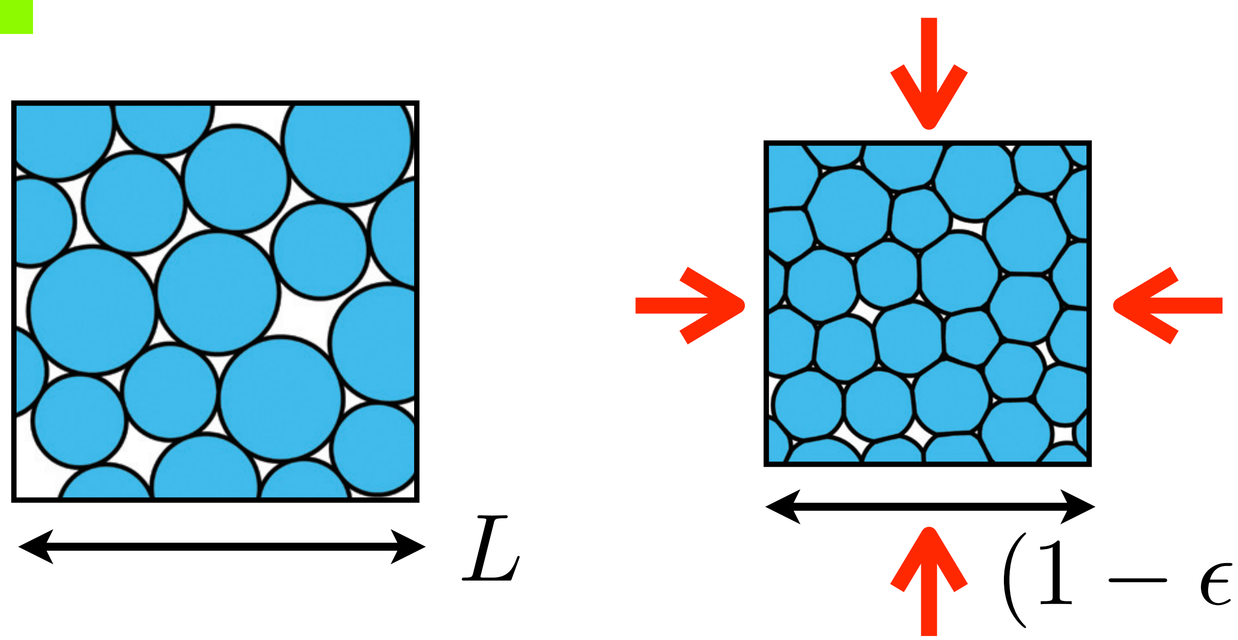


force f

dimensionless overlap $\delta = 1 - \frac{|\vec{r}_j - \vec{r}_i|}{R_i + R_j}$

$f \sim \delta^\alpha$

Global



L

$(1 - \epsilon)L$

(exaggerated)

pressure p

strain

$$\epsilon = \frac{\Delta V}{V} = \frac{\Delta \phi}{\phi}$$

‘Trivial’ or ‘Naive’ Scaling

Local = **Global**

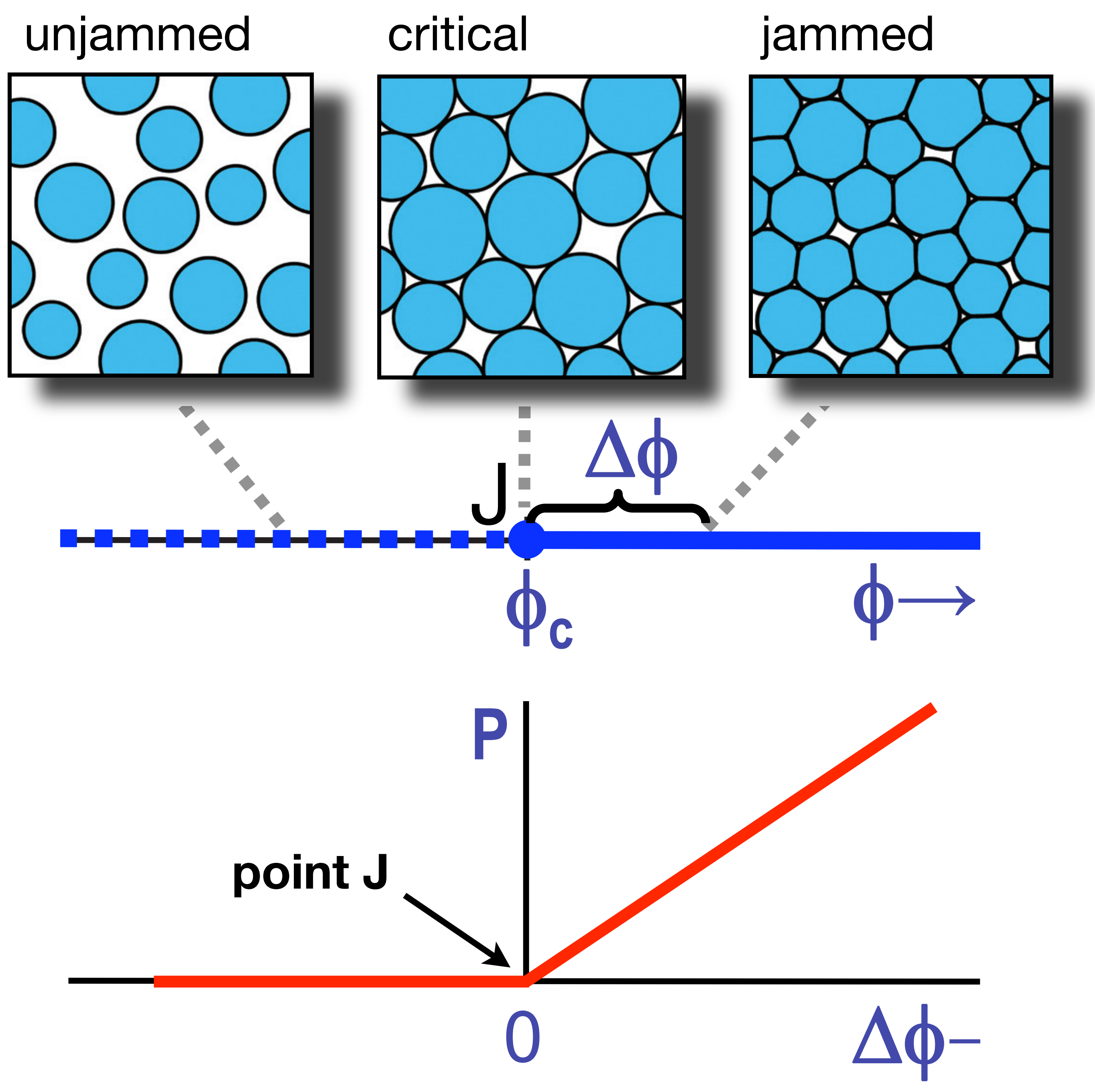
$$f \sim \delta^{\alpha}$$

$$f \sim p$$

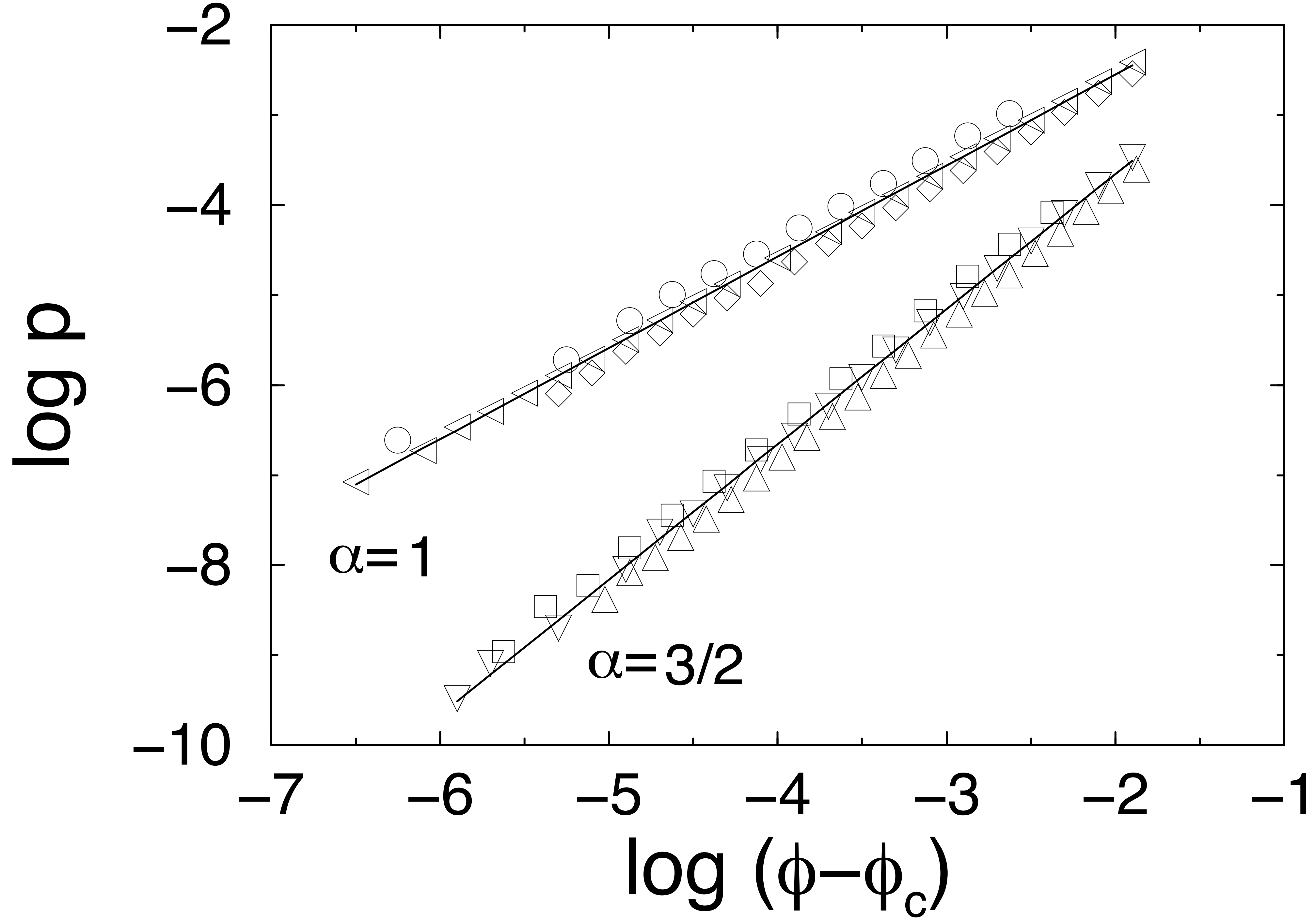
$$\delta \sim \Delta\phi$$

$$p \sim \Delta\phi^{\alpha}$$

Pressure Scaling

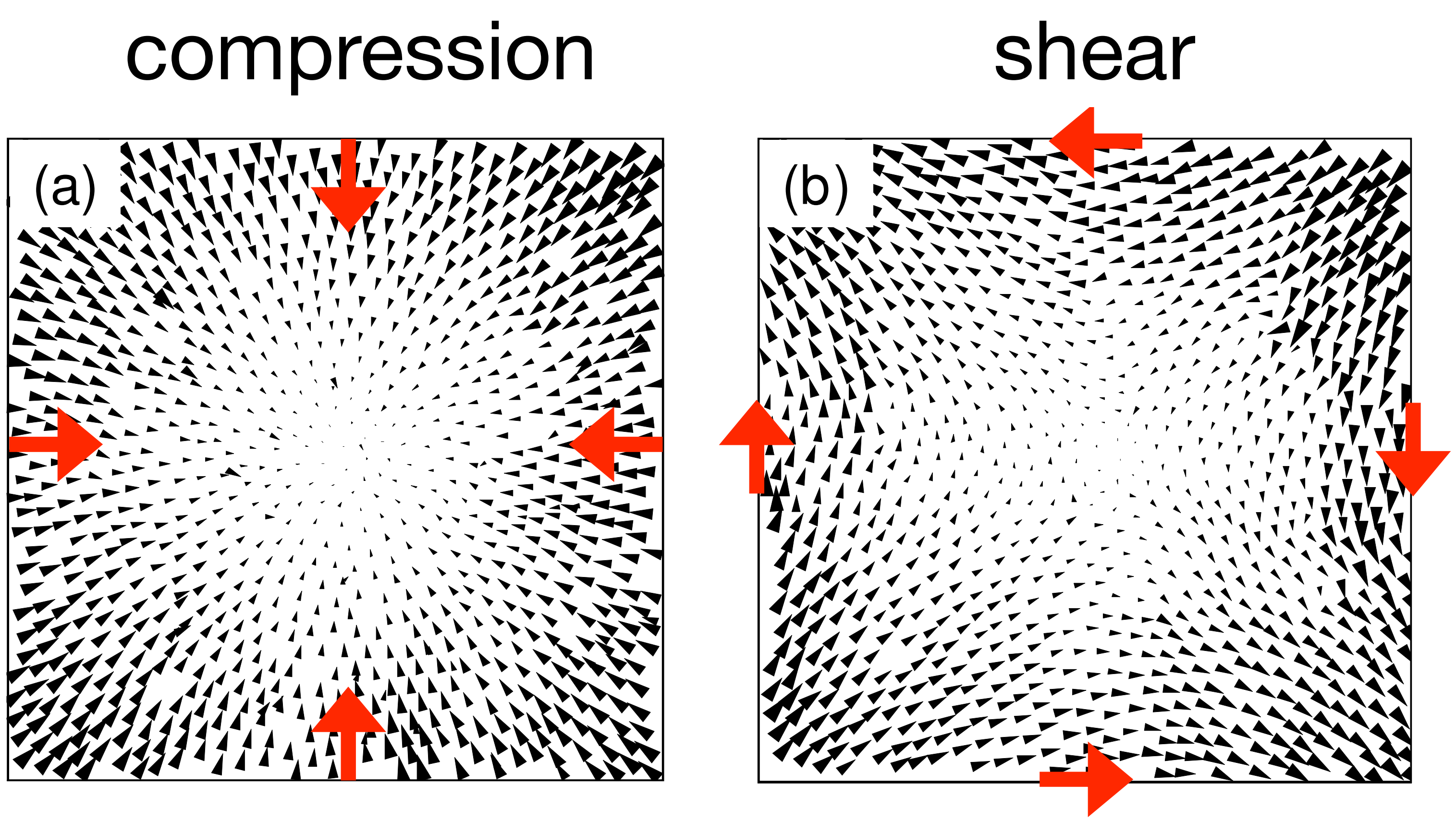


Pressure Scaling



O'Hern et al, PRE 2002

Mechanical Deformations

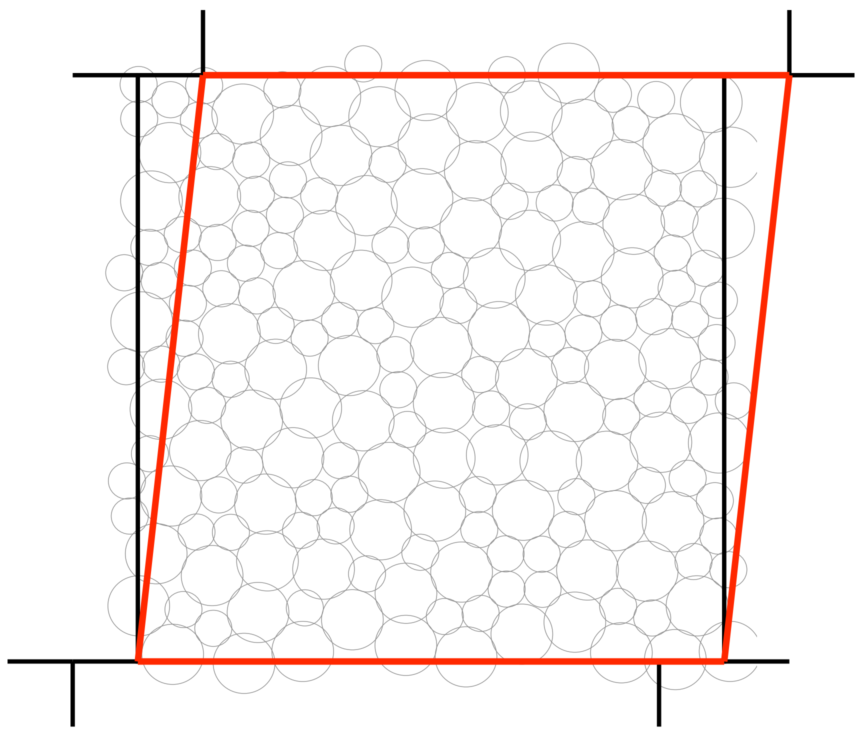


affine
smooth
continuum-like
far from point J

non-affine
disordered
obviously discrete
near point J

Affine: Going Halfway

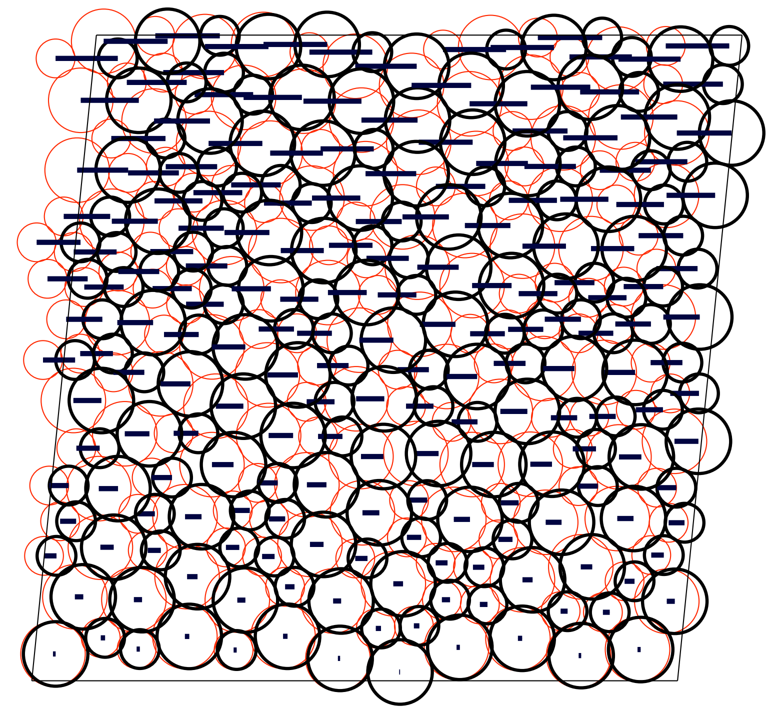
illustration:
C. Maloney, CMU



simple shear:

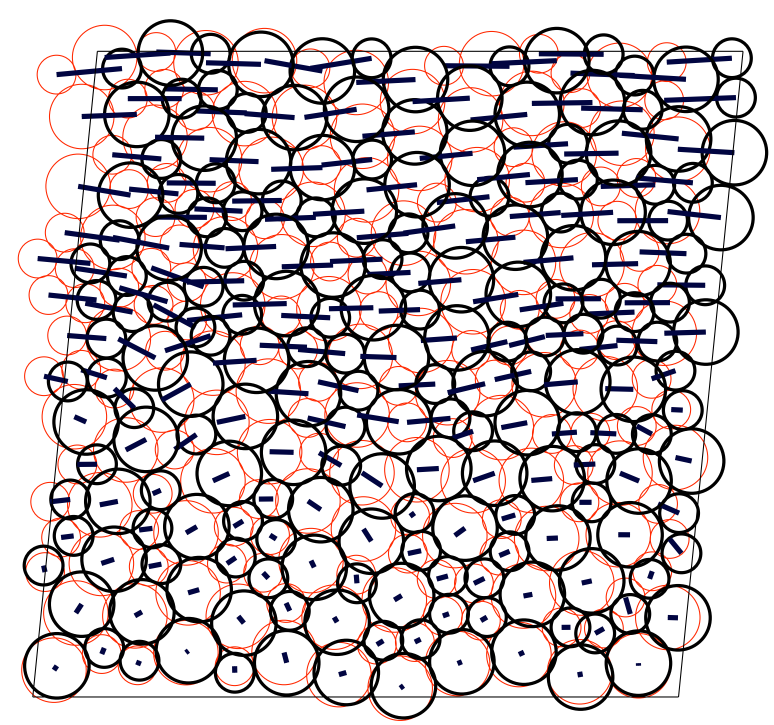
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + \epsilon y \\ y \end{pmatrix}$$

(i)



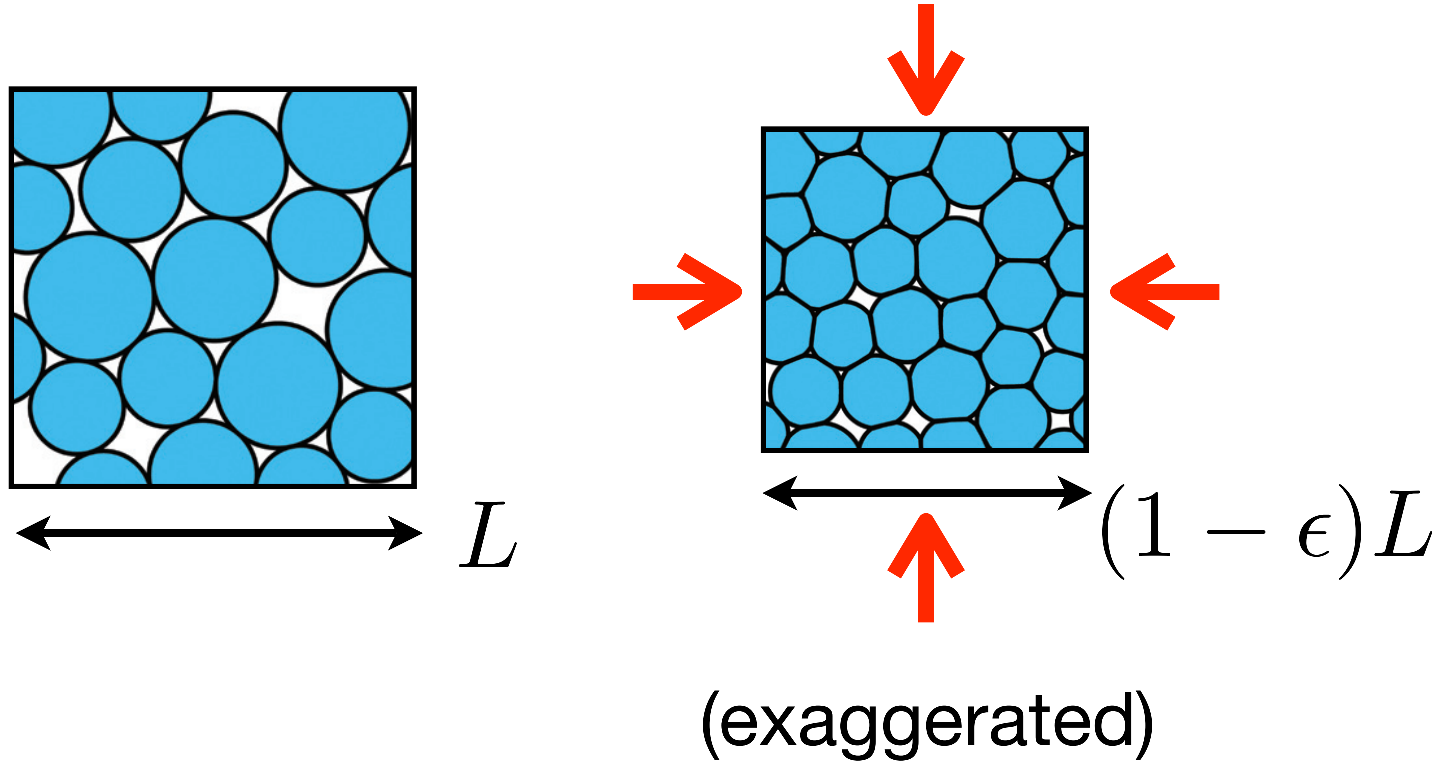
affine

(ii)



relax

Mechanics: Bulk modulus K

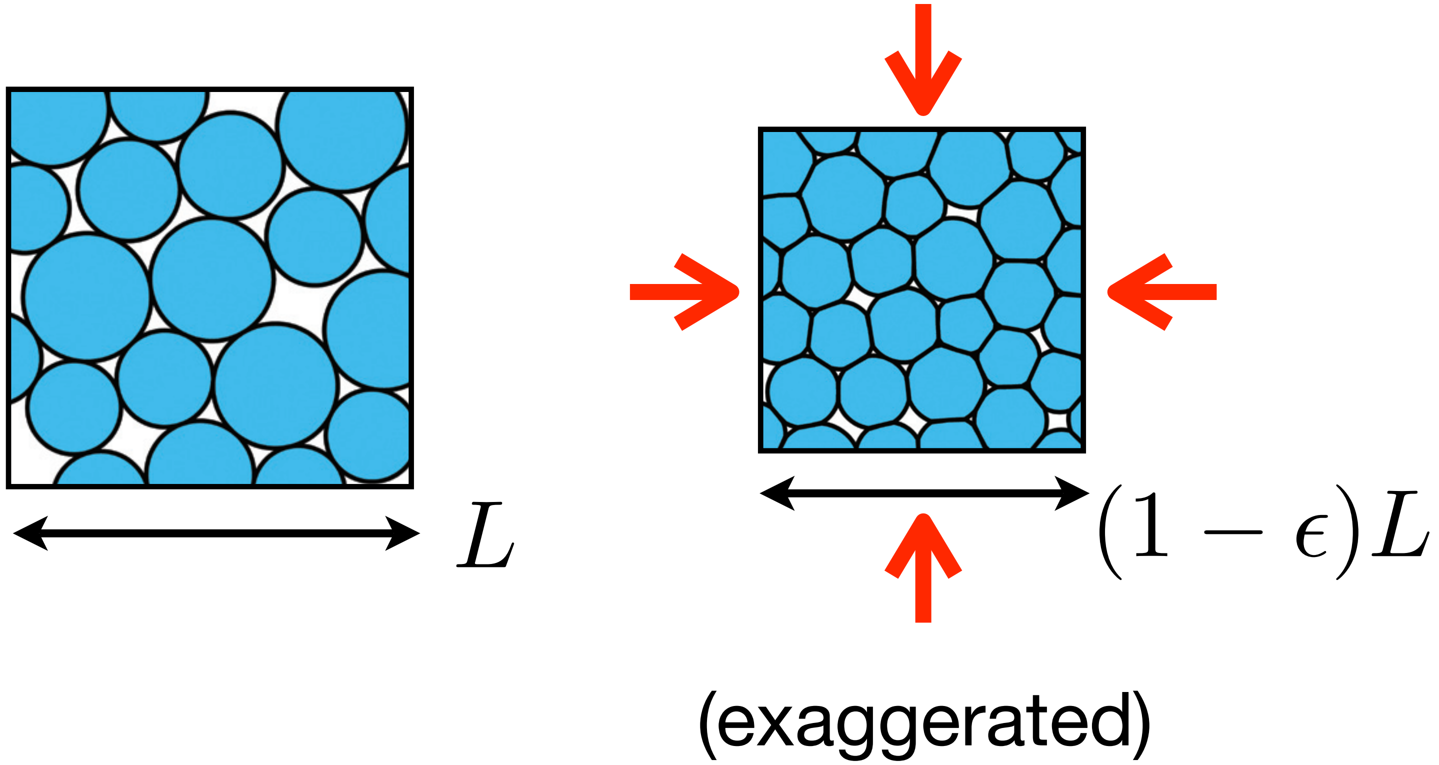


bulk modulus

$$U = \frac{1}{2} K \epsilon^2$$

$$K = \frac{\partial^2 U}{\partial \epsilon^2}$$

Mechanics: Bulk modulus K



bulk modulus

$$U = \frac{1}{2} K \epsilon^2$$

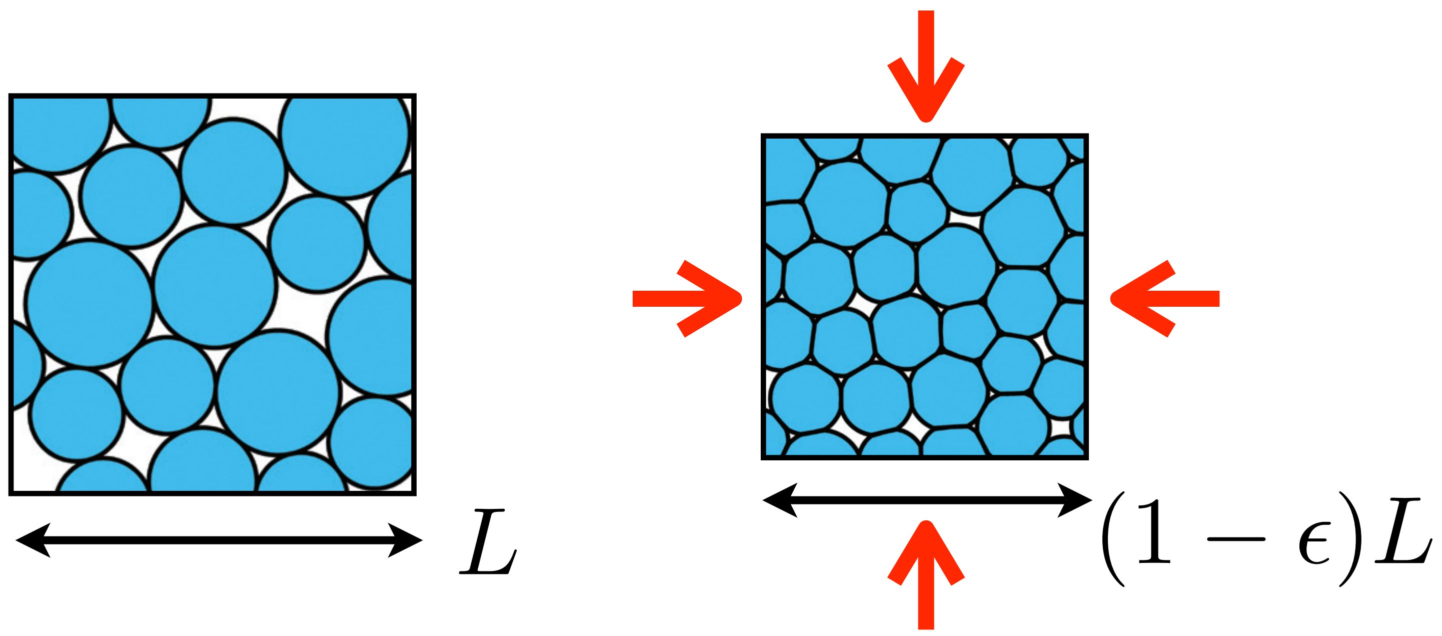
$$K = \frac{\partial^2 U}{\partial \epsilon^2}$$

$$U \sim f\delta \sim \delta^{\alpha+1}$$

affine approx $\Rightarrow \; \epsilon \sim \delta \sim \Delta\phi$

$$K \simeq \Delta\phi^{\alpha-1}$$

Mechanics: Bulk modulus K



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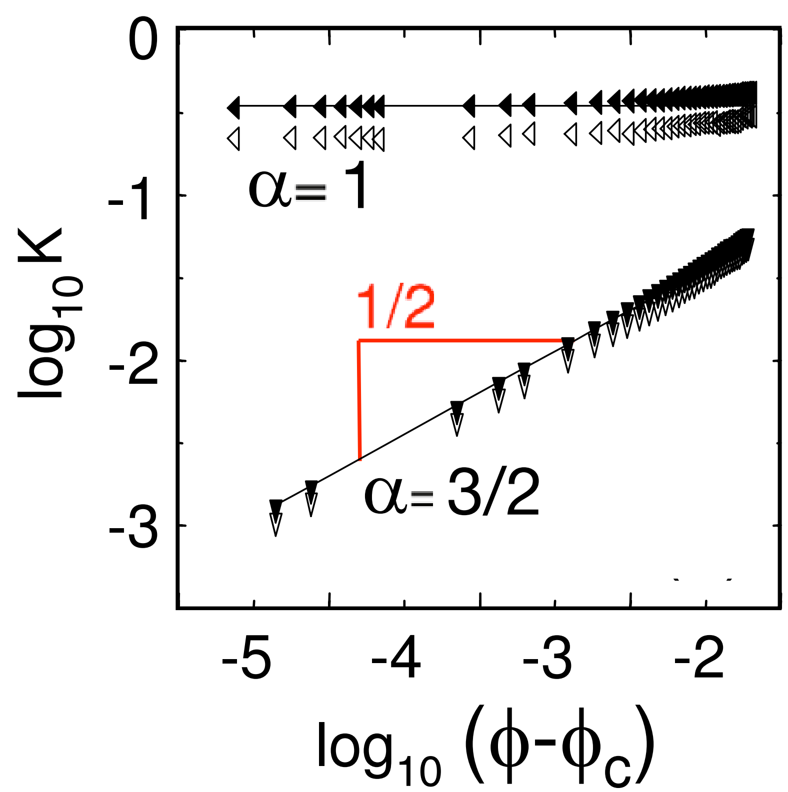


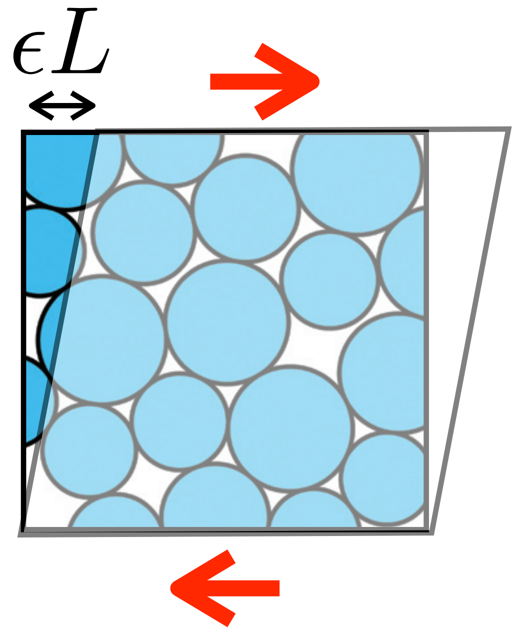
O'Hern et al, PRE 2002

it worked!

…but we will see this was lucky

$$K \simeq \Delta\phi^{\alpha-1}$$

Mechanics: Shear modulus G

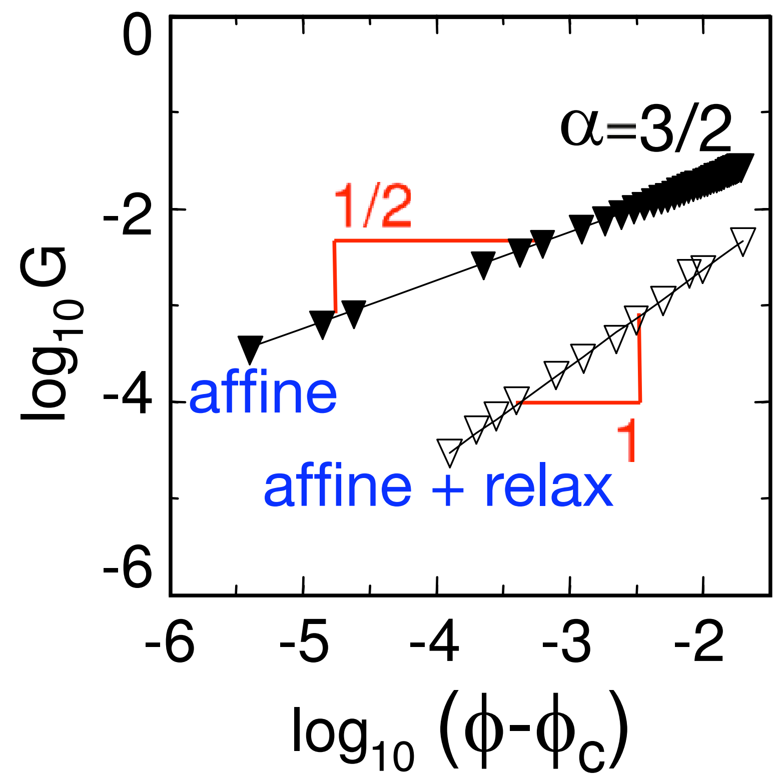


shear modulus

$$U = 2G\epsilon^2$$

simple guess (yet again): deformation is affine, local motion can be inferred from global

⇒ shear should be just like compression



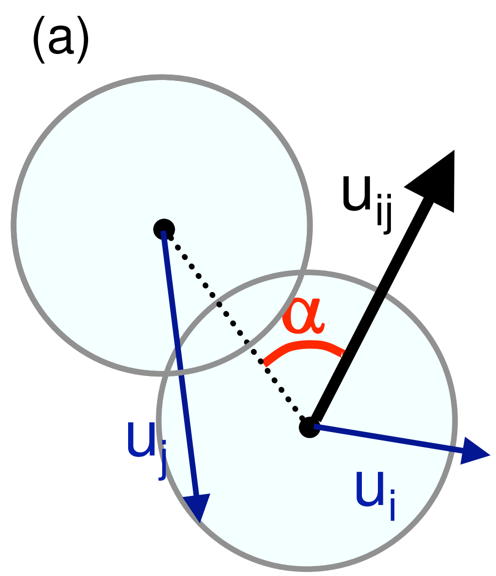
O'Hern et al, PRE 2002

NO! completely wrong!

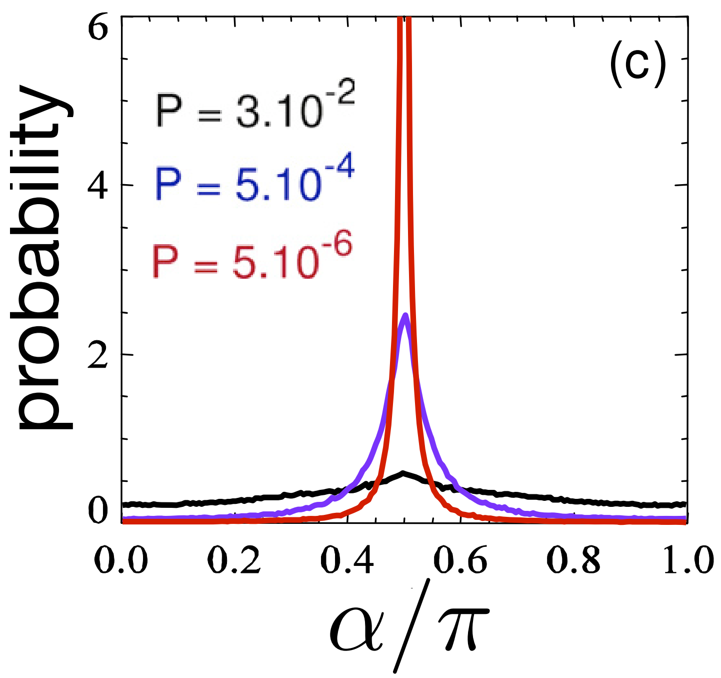
$$G \sim \Delta\phi^{\alpha - \frac{1}{2}}$$

Mechanics: Shear modulus G

What did we get wrong?



shear

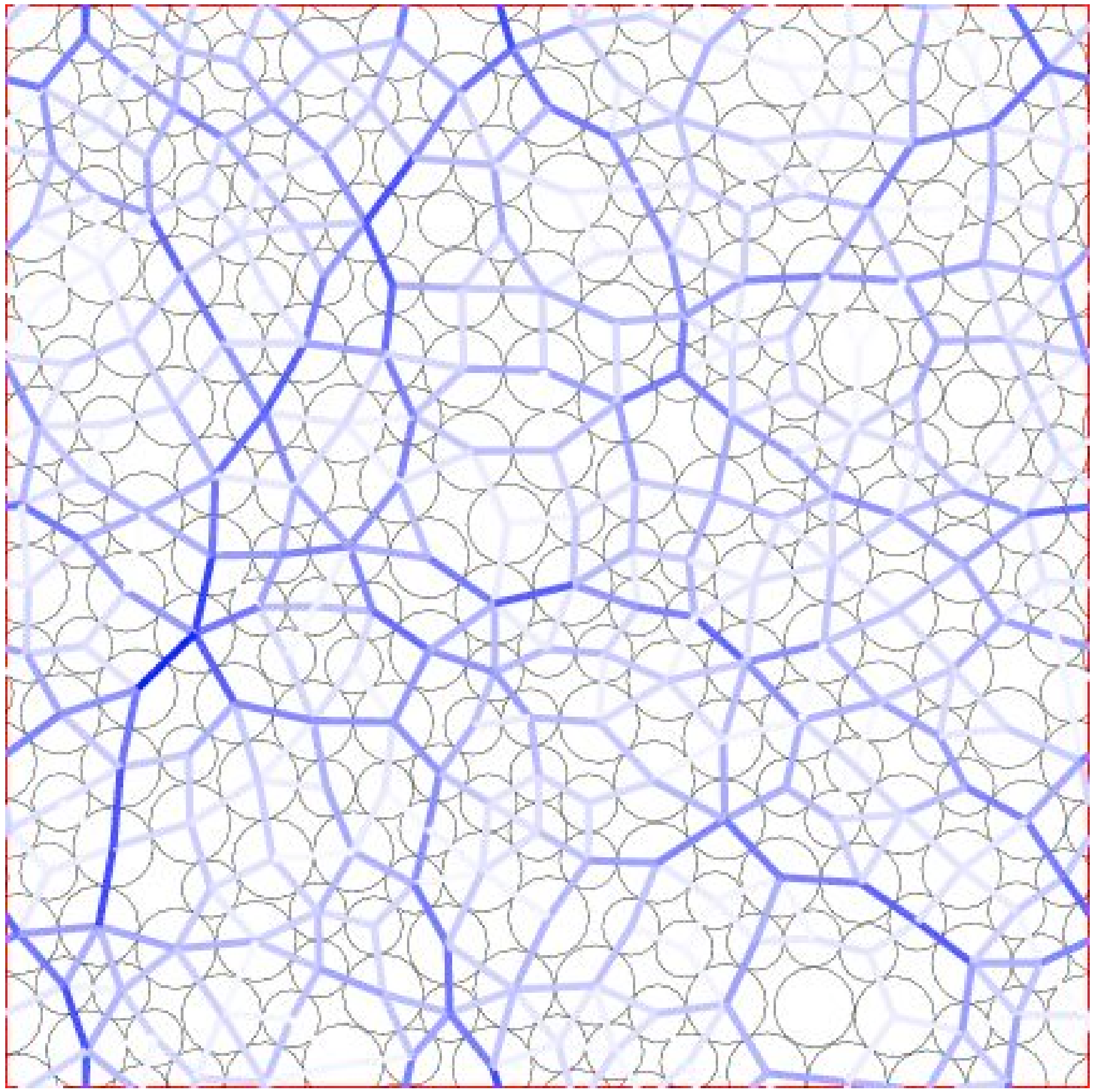


peak: motion not affine

we were just lucky!

Ellenbroek PRL 2006

Geometry: Counting contacts



O'Hern group, Yale

Frictionless spheres at Point J

z	avg # contacts per grain
N	grains
$\frac{1}{2}zN$	contacts (and contact forces)

"**kissing constraint**"

$$|\vec{r}_i - \vec{r}_j| = R_i + R_j$$

$$z \leq 2D$$

force balance

$$\sum_j \vec{f}_{ij} = 0$$

$$z \geq 2D$$

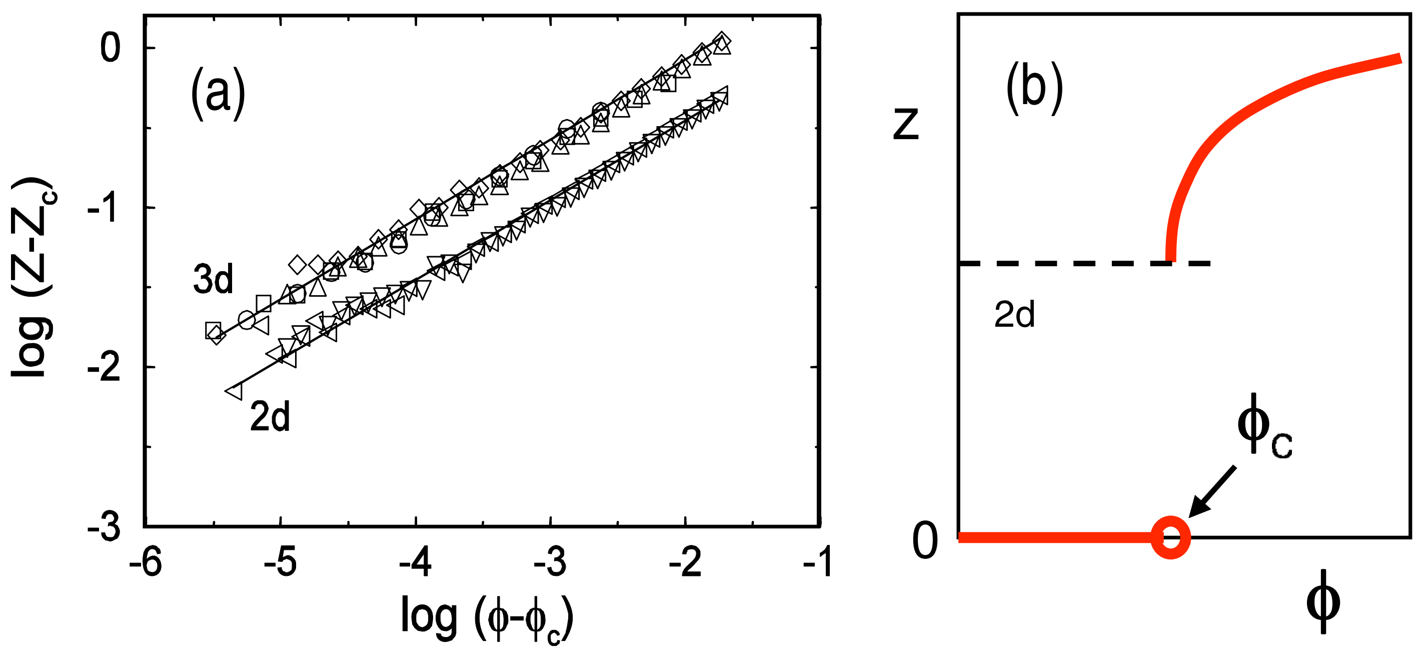
Geometry: Counting contacts

how should z depend on distance to Point J?

simple guess: compression is like inflating grain radii
(**affine** approximation)
⇒ close gaps ⇒ make new contacts

would give...

$\Delta z \sim \Delta \phi$ **but NO!** $\Delta z \sim \sqrt{\Delta \phi}$



O'Hern et al, PRE 2002

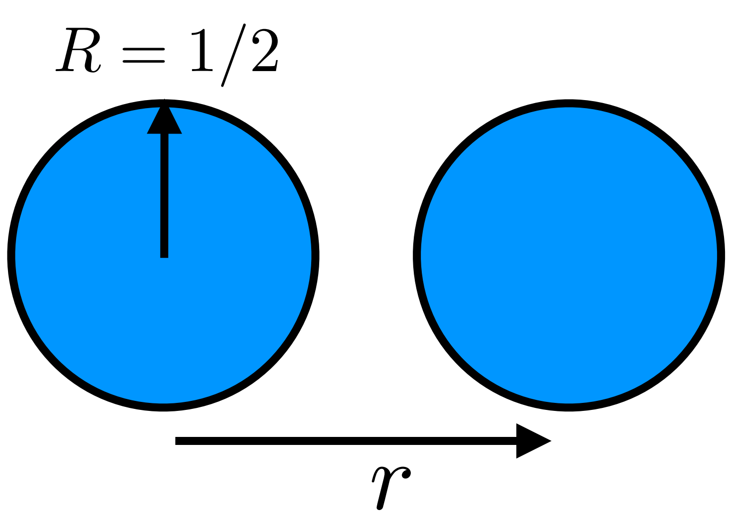
Geometry: Counting contacts

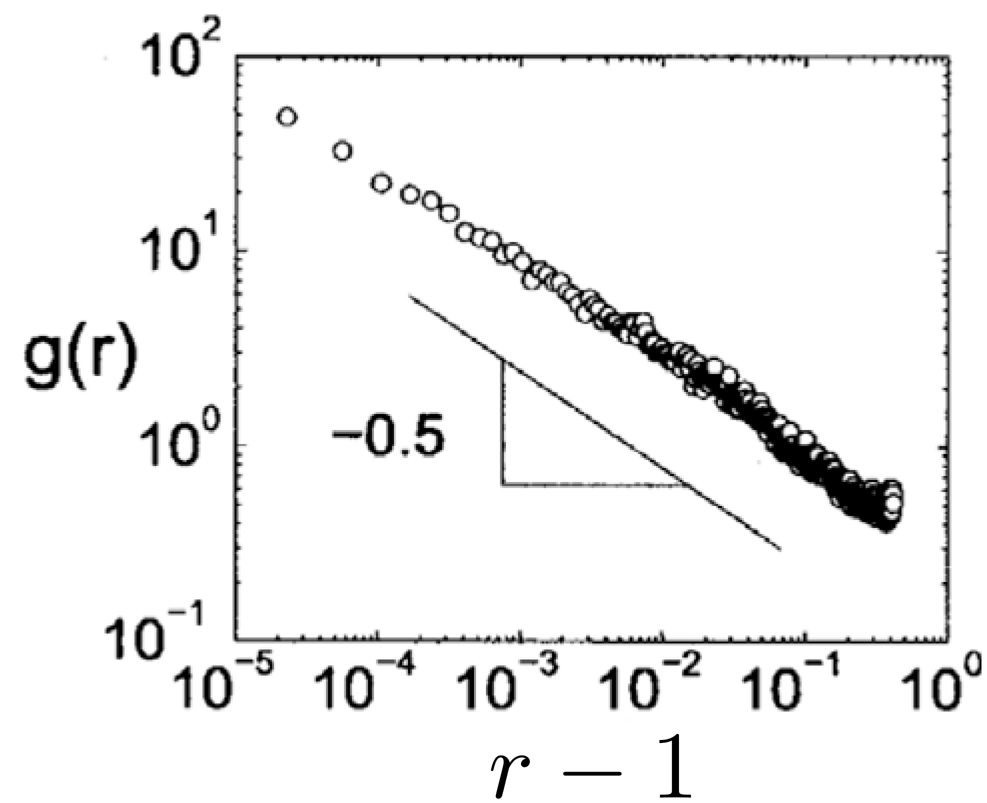
chicken, meet egg

what went wrong?

close gaps ⇒ make new contacts $$\frac{\partial z}{\partial \delta} \sim g(1+\delta)$$

…assumes prob(gap size) ~ const… but it's **not**





$$g(r)\, r^d \mathrm{d}\Omega\, \mathrm{d}r$$ prob. of finding another grain centered within $\mathrm{d}r$ of r

O'Hern et al, PRE 2002

Summary: Nontrivial Scalings

geometry

excess contacts:

$$\Delta z \sim \sqrt{\Delta \phi}$$

divergence in radial dist. fn.:

$$g(r) \sim \frac{1}{\sqrt{r-1}}$$

elasticity

bulk modulus:

$$K \sim \Delta \phi^{\alpha - 1}$$

shear modulus:

$$G \sim \Delta \phi^{\alpha - 1/2}$$

ratio: $$\frac{G}{K} \sim \Delta z$$

Vibrations: Density of States

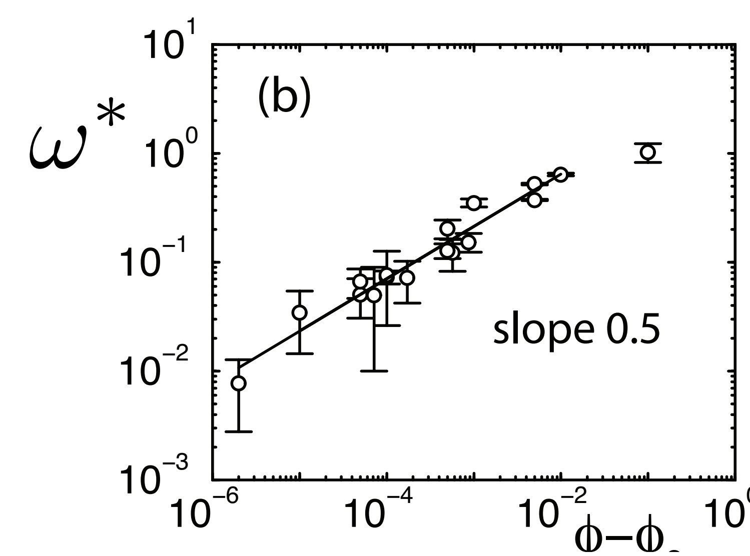
write down eom for each grain
⇒ matrix equation

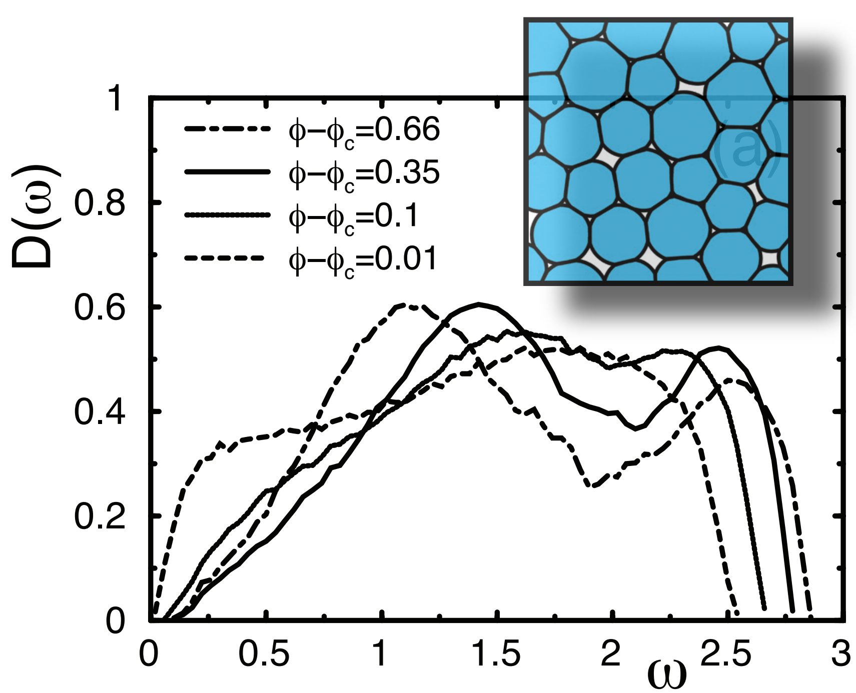
$$
\begin{aligned}
-\hat{\mathcal{K}}\mathbf{u} + \mathbf{F}_{\text{ext}} &= m\ddot{\mathbf{u}} \\
&= m\omega^2\mathbf{u}
\end{aligned}
$$

↑ displacement

no ext. force ⇒ eigenvalue eqn

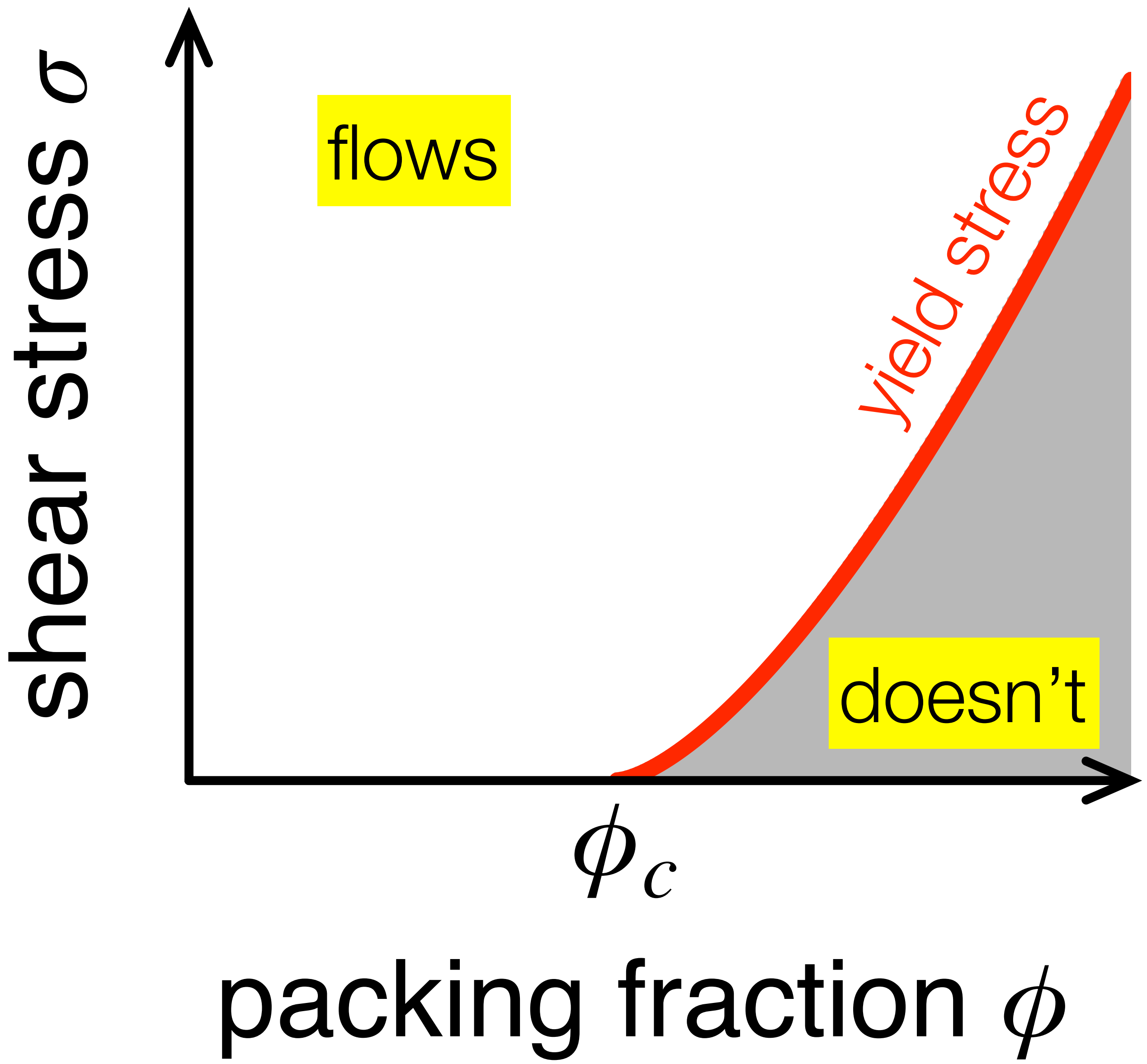
density of states: histogram of eigenfrequencies

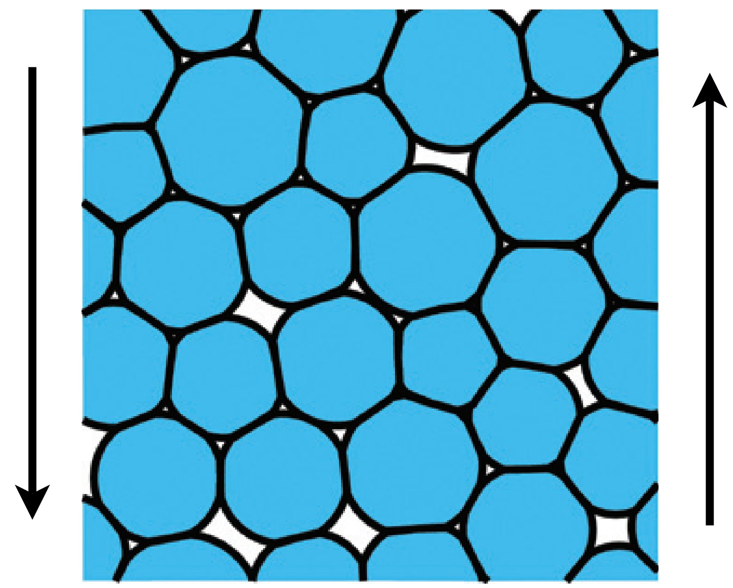




figures from review article by Liu, Wyart, van Saarloos and Nagel, 2010

Beyond Linear Response: Flow

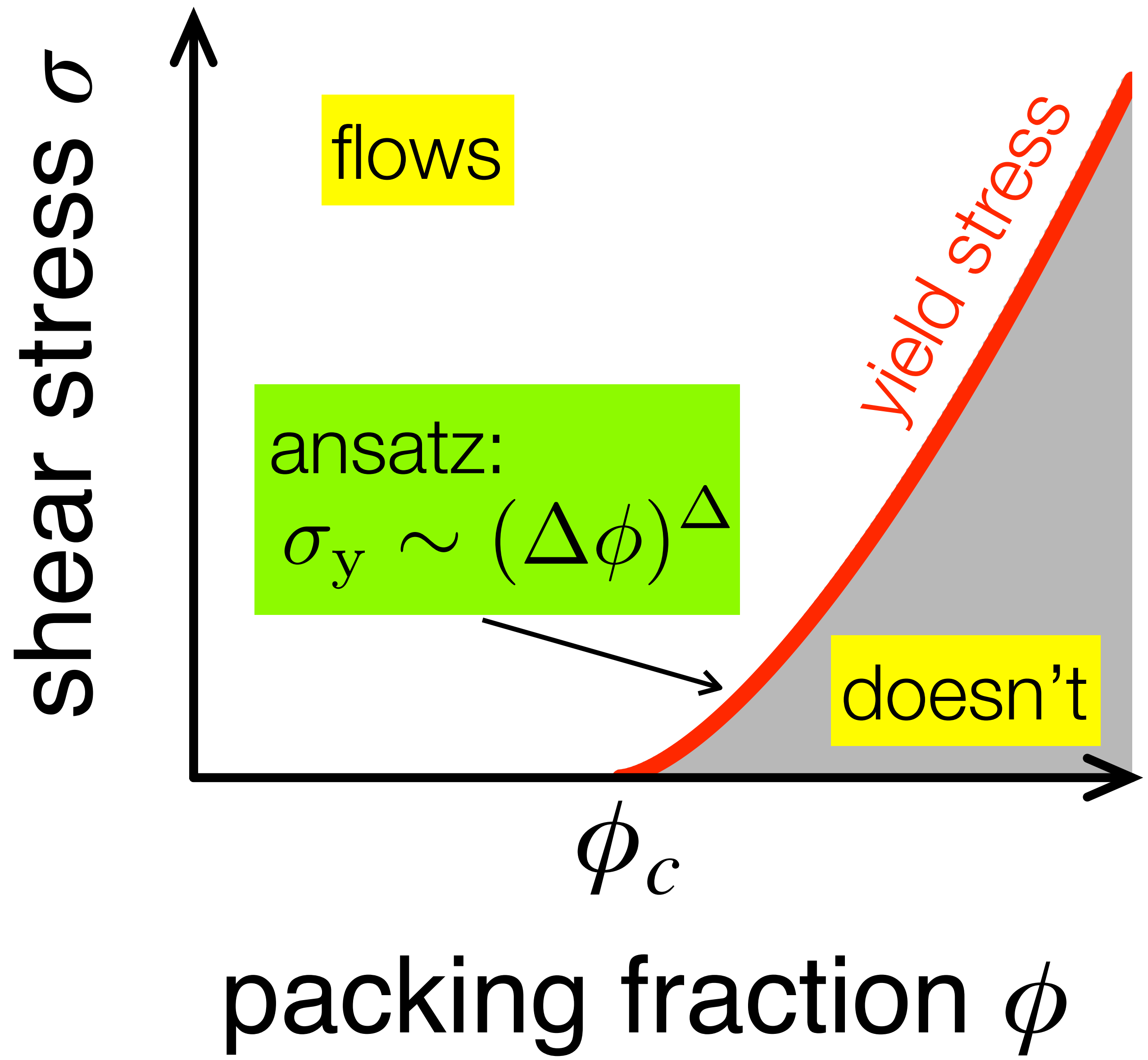


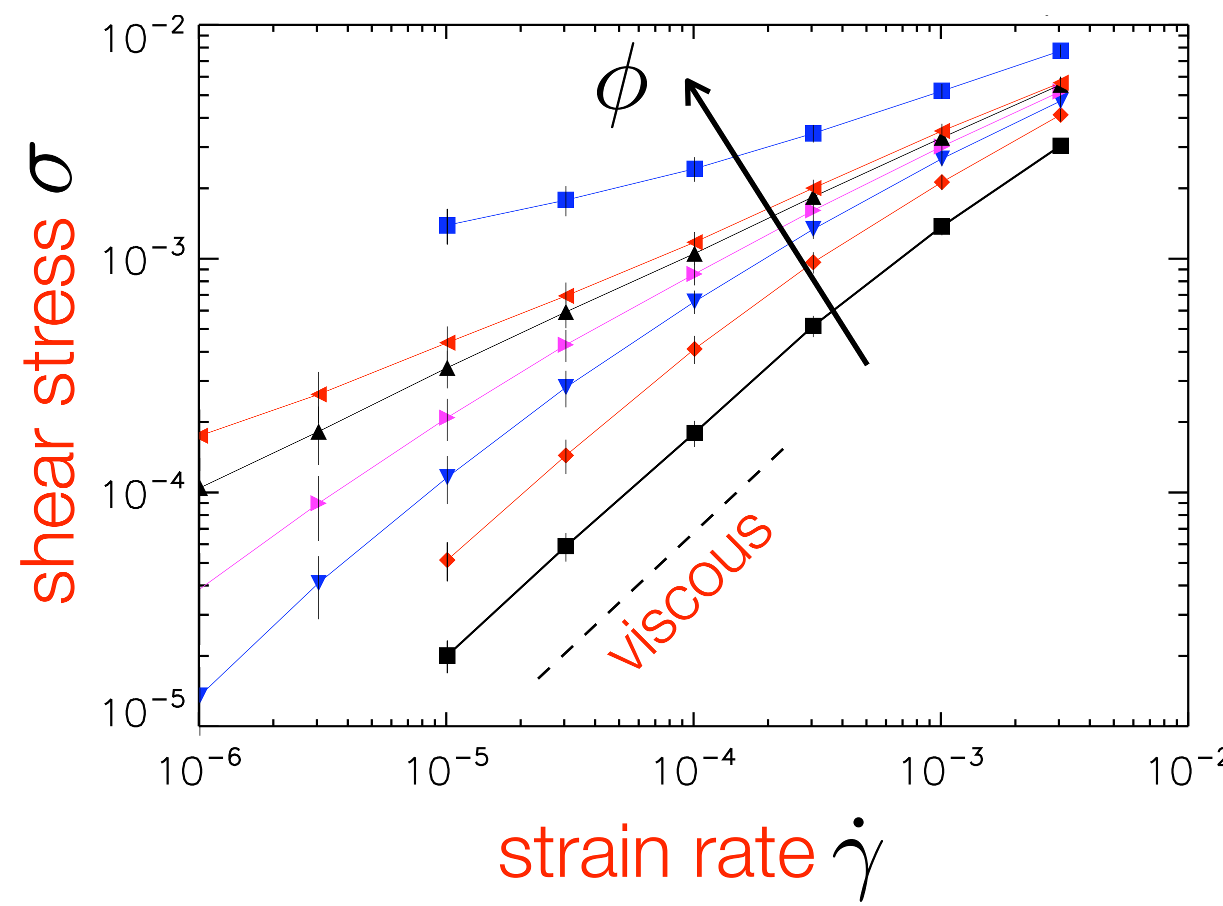


impose strain rate $\dot{\gamma}$

measure shear stress σ

Beyond Linear Response: Flow





strain softening

evident yield stress?

data: Tighe, Woldhuis, Remmers, van Saarloos and van Hecke

Beyond Linear Response: Flow

empirical fact:

many soft matter systems obey **Herschel-Bulkley rheology**

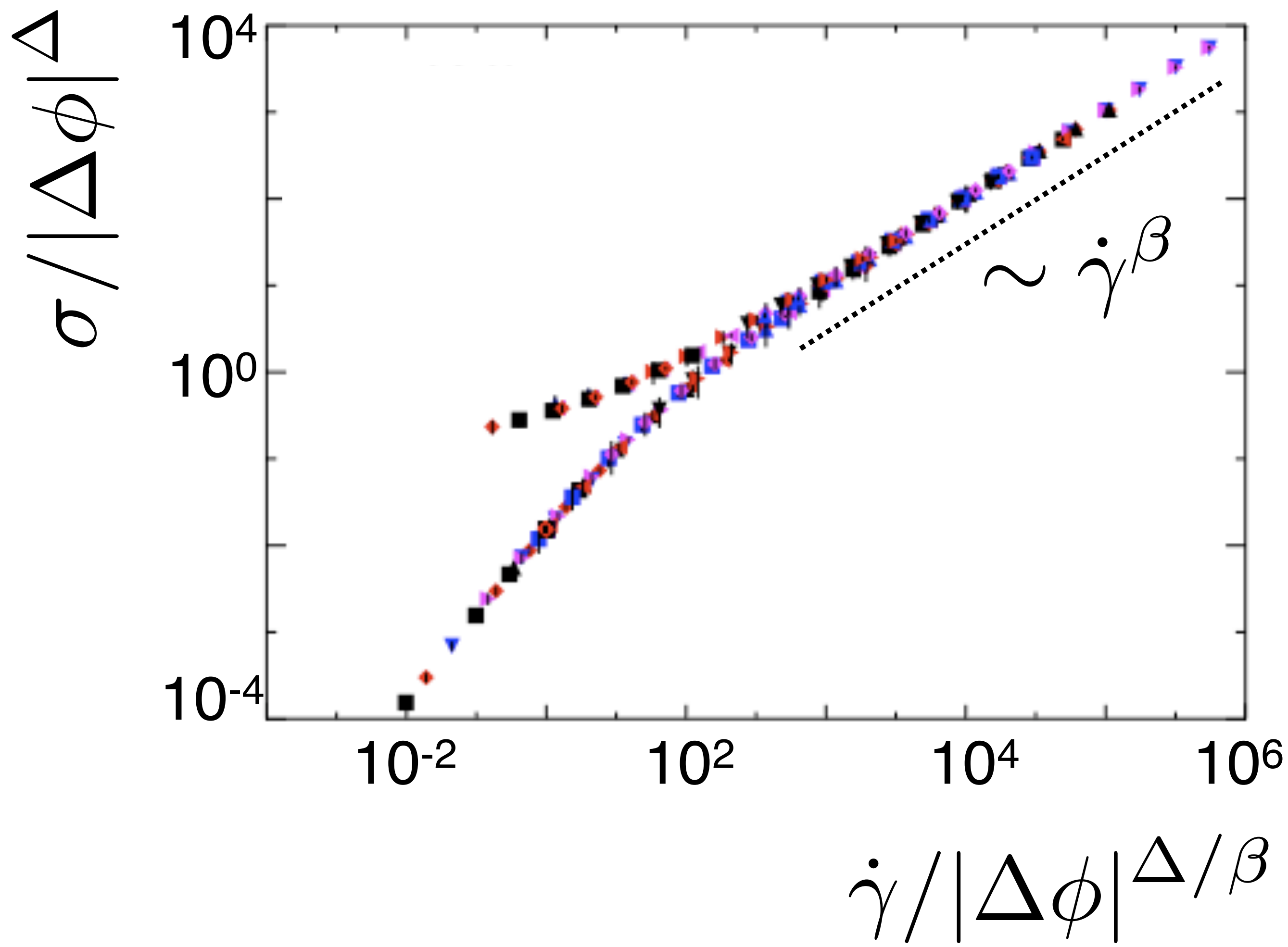
$$\sigma = \sigma_{\mathrm{y}} + A\dot{\gamma}^{\beta}$$

combine with $\sigma_{\mathrm{y}} \sim (\Delta\phi)^{\Delta}$

$$\frac{\sigma}{(\Delta\phi)^{\Delta}} = 1 + A\left(\frac{\dot{\gamma}}{(\Delta\phi)^{\Delta/\beta}}\right)^{\beta}$$

suggests **rescaled** coordinates

Beyond Linear Response: Flow



$\phi_c \approx 0.841$

$\Delta \approx 1.5$

$\beta \approx 0.5$

see also:

Olsson & Teitel PRL 2007
Hatano JPSJ 2008, PRE 2009
Tighe et al., in prep.

10^3 bubbles

5 decades in $\dot{\gamma}$

data: Tighe, Woldhuis, Remmers, van Saarloos and van Hecke

Why study jamming?

⇒ (hope for) **universality**

many properties governed by one attribute: **distance to transition**

some materials (or models) more **convenient** than others

not so much:
force law, friction, particle shape all matter

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kind of…
geometry and elasticity ~ distance to isostaticity

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definitely!
first experiments on "frictionless spheres" are turning up now…
foams, emulsions, etc.

Useful References

van Hecke
J. Phys. Cond Mat. 2010

O'Hern et al. "Epitome of disorder" paper
PRE 2002

Wyart et al.
EPL 2005