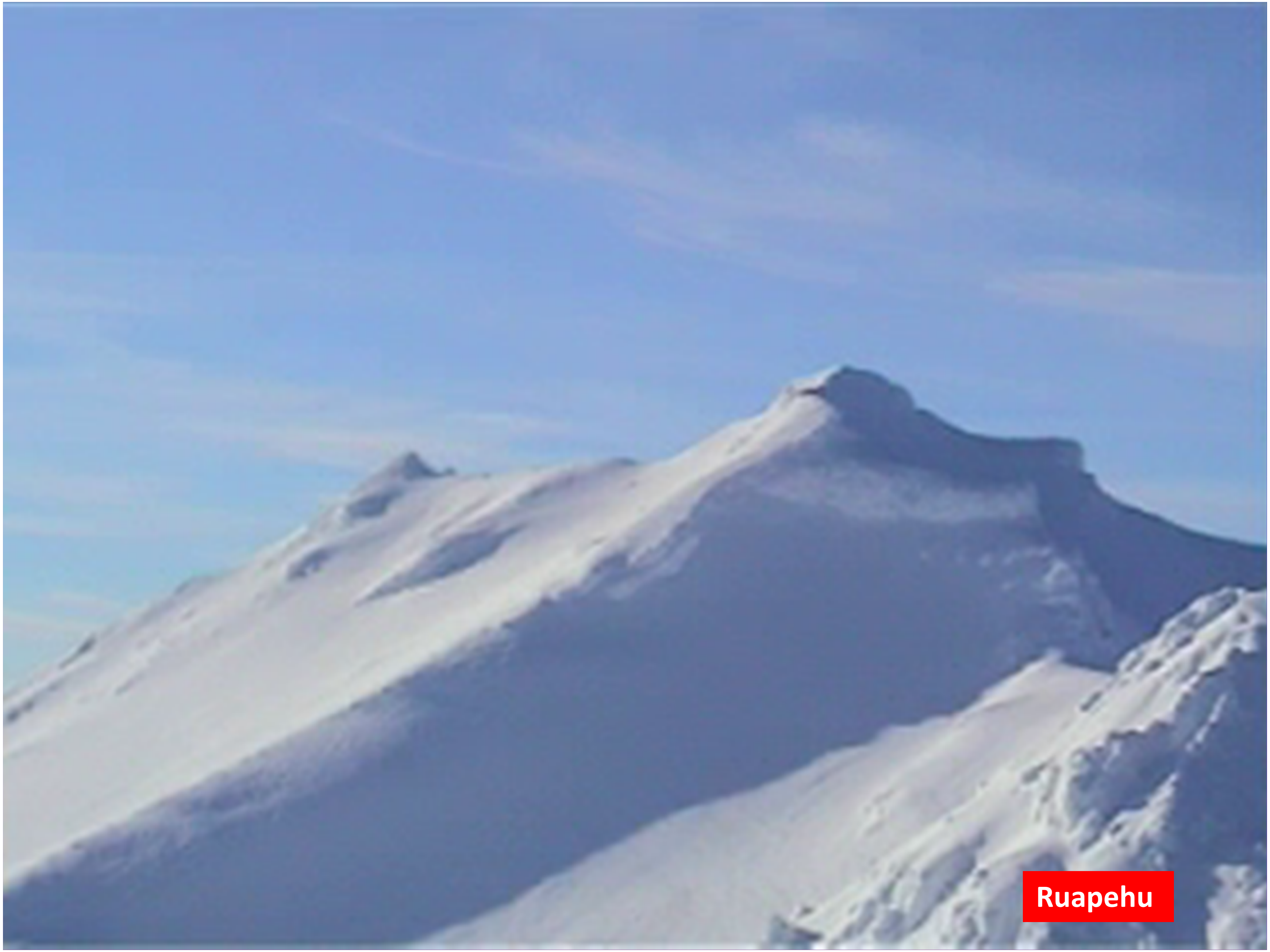
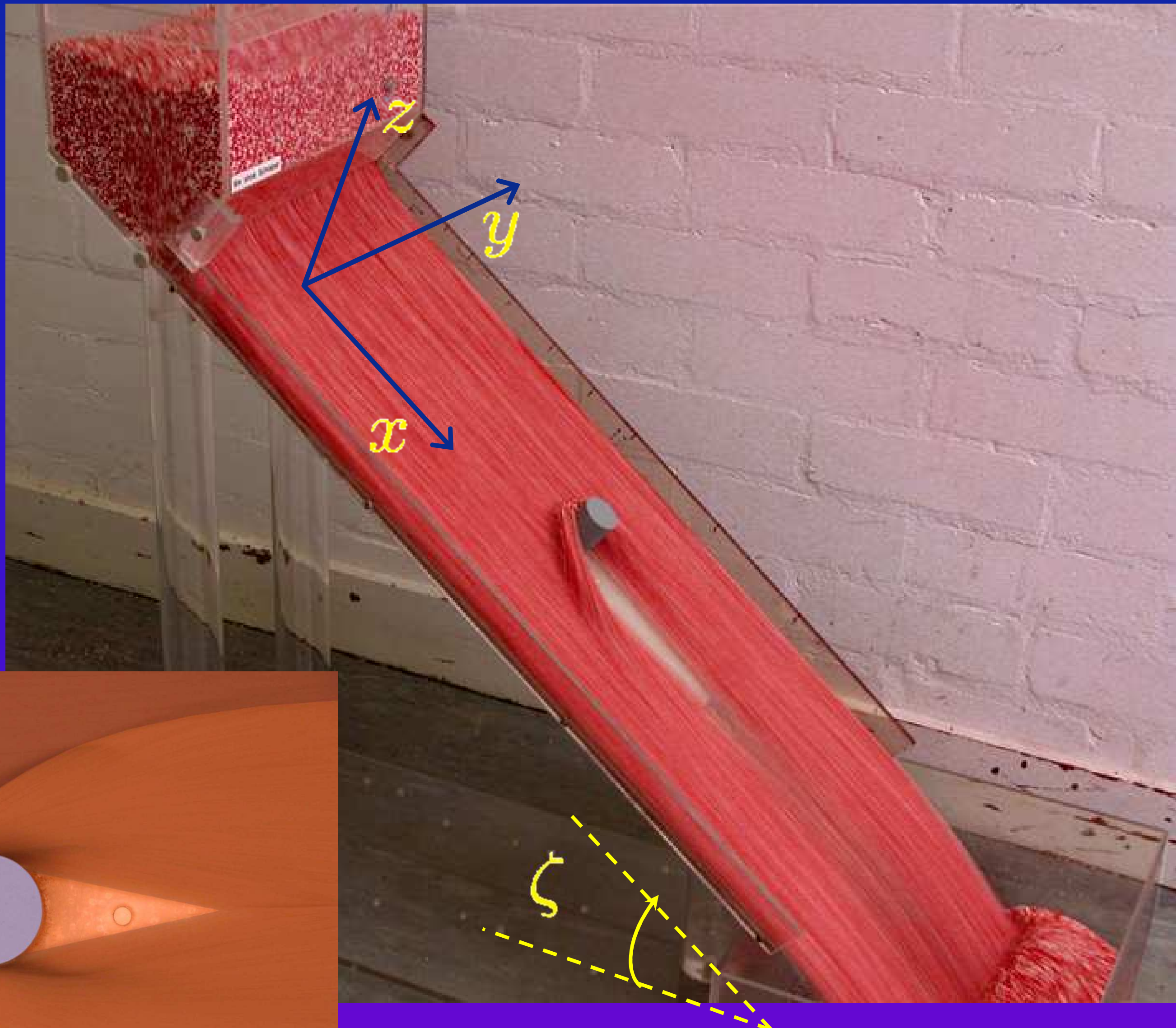
Granular Avalanches

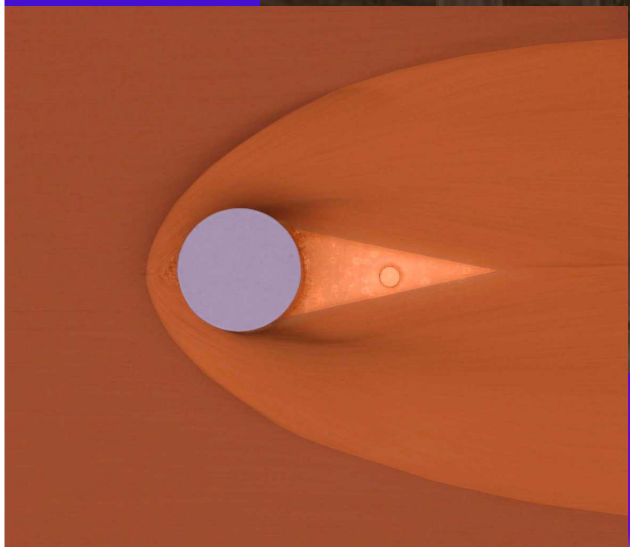
Nico Gray, School of Mathematics, The University of Manchester

Mount St Helens May 18, 1980


Ruapehu

Experimental chute setup and coordinate system





Cui & Gray *J. Fluid. Mech.*, in press.

Derivation of the depth-averaged equations

- Mass and momentum balances

$$\nabla \cdot \boldsymbol{u} = 0$$
$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g},$$

- assume density ρ constant
- bulk velocity $\boldsymbol{u} = (u, w)^T$ and $\otimes$ is the dyadic product
- stress $\boldsymbol{\sigma}$ split into a pressure p and a deviatoric part $\boldsymbol{\tau}$

$$\sigma = -p\mathbf{1} + \boldsymbol{\tau}$$

- subject to kinematic conditions at surface and base

$$\frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} - w = 0, \quad \text{at} \quad z = s(x,t) \quad \text{and} \quad z = b(x,t),$$

- and surface and basal traction conditions

$$\begin{aligned} z = s(x,t) : & \qquad \boldsymbol{\sigma n} = \mathbf{0}, \\ z = b(x,t) : & \qquad \boldsymbol{\sigma n} = -(\boldsymbol{u}/|\boldsymbol{u}|)\mu(\boldsymbol{n} \cdot \boldsymbol{\sigma n}) + \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\sigma n}), \end{aligned}$$

where $\boldsymbol{n}$ is the normal and μ is the friction coefficient.

- integrate $\nabla \cdot \boldsymbol{u} = 0$ through depth using Leibniz' Rule

$$\frac{\partial}{\partial \lambda} \int_{b(\lambda)}^{s(\lambda)} f \, dz = \int_{b(\lambda)}^{s(\lambda)} \frac{\partial f}{\partial \lambda} \, dz + \left[f \frac{\partial z}{\partial \lambda} \right]_{b(\lambda)}^{s(\lambda)},$$

- to exchange the order of integration and differentiation

$$\int_{b(x,t)}^{s(x,t)} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = \frac{\partial}{\partial x} \left(\int_{b(x,t)}^{s(x,t)} u \, dz \right) - \left[u \frac{\partial z}{\partial x} - w \right]_{b(x,t)}^{s(x,t)}.$$

- Defining the depth-averaged velocity and thickness

$$\bar{u} = \frac{1}{h} \int_b^s u \, dz, \qquad h(x,t) = s(x,t) - b(x,t)$$

- and using the kinematic boundary conditions the depth-averaged mass balance becomes

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = 0.$$

- Making the shallowness approximation
- the normal momentum balance and the surface traction condition imply that the pressure p is lithostatic

$$p = \rho g (s - z) \cos\zeta$$

- depth-averaging the downslope momentum balance

$$\rho\left(\frac{\partial}{\partial t}(h\overline{u}) + \frac{\partial}{\partial x}(h\overline{u^2})\right) - \left[\rho u\left(\frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} - w\right)\right]_b^s$$
$$= \rho g h \sin\zeta + \frac{\partial}{\partial x}(h\overline{\sigma_{xx}}) - \left[\sigma_{xx}\frac{\partial z}{\partial x} - \sigma_{xz}\right]_b^s .$$

- Using the kinematic condition, approximating the basal traction, neglecting $\overline{\tau}_{xx}$ and assuming $\overline{u^2} = \overline{u}^2$

$$\frac{\partial}{\partial t}(h\overline{u}) + \frac{\partial}{\partial x}(h\overline{u}^2) + \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\cos\zeta\right)$$
$$= hg\cos\zeta(\tan\zeta - \mu(\bar{u}/|\bar{\boldsymbol{u}}|)) - hg\frac{\partial b}{\partial x}\cos\zeta$$

- finally the equations are non-dimensionalized

Two-dimensional depth-averaged system

- For avalanche thickness h and mean velocity $\bar{\boldsymbol{u}} = (\bar{u}, \bar{v})$ in the downslope x and cross-slope y directions.

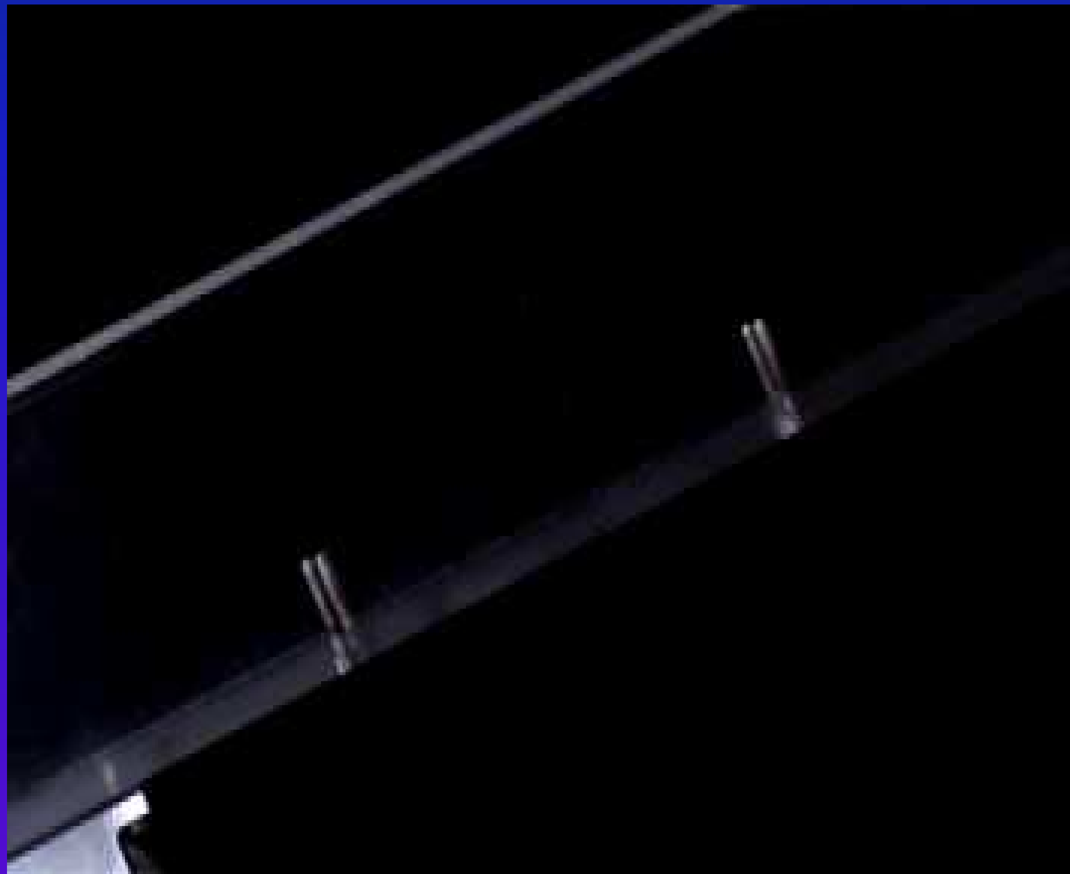
$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) &= 0, \\
\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(h\bar{u}^2) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) + \frac{\partial}{\partial x}\left(\frac{1}{2}h^2\cos\zeta\right) &= hS_{(x)}, \\
\frac{\partial}{\partial t}(h\bar{v}) + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}(h\bar{v}^2) + \frac{\partial}{\partial y}\left(\frac{1}{2}h^2\cos\zeta\right) &= hS_{(y)},
\end{aligned}
$$

- source terms composed of gravity, basal friction μ and gradients of the basal topography b

$$
\begin{aligned}
S_{(x)} &= \sin\zeta - \mu(\bar{u}/|\bar{\boldsymbol{u}}|)\cos\zeta - \frac{\partial b}{\partial x}\cos\zeta, \\
S_{(y)} &= - \mu(\bar{v}/|\bar{\boldsymbol{u}}|)\cos\zeta - \frac{\partial b}{\partial y}\cos\zeta,
\end{aligned}
$$

Grigorian *et al.* 1967; Gray *et al.* P. Roy. Soc. 1999, JFM 2003

Upslope propagating granular bores



- observations suggest a shock separating constants states

$$
\begin{aligned}
x < \xi: \qquad & h(x,t) = h_1, \;\; \bar{u}(x,t) = \bar{u}_1, \\
x > \xi: \qquad & h(x,t) = h_2, \;\; \bar{u}(x,t) = \bar{u}_2,
\end{aligned}
$$

- At shocks the mass and momentum jump conditions are

$$\begin{aligned} [\![h(\bar{u} - v_n)]\!] &= 0, \\ [\![h\bar{u}(\bar{u} - v_n)]\!] + [\![\tfrac{1}{2} h^2 \cos\zeta]\!] &= 0, \end{aligned}$$

- where v_n is the normal propagation speed and $[\![\cdot]\!]$ is the jump across the discontinuity.
- Assuming the grains come to rest after a bore

$$v_n = -\sqrt{\frac{h_1}{h_2}\left(\frac{h_1 + h_2}{2}\right)\cos\zeta}.$$

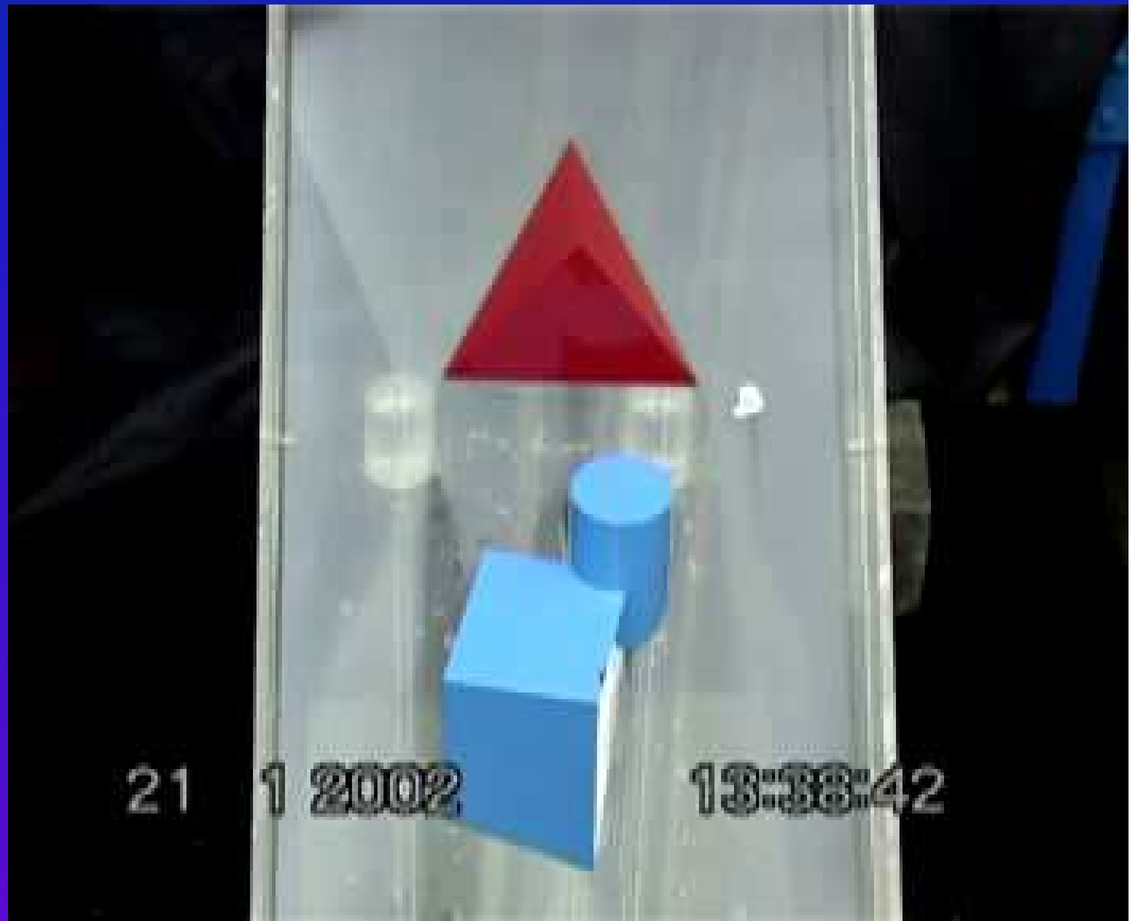
- In the lab experiments

$$h_1 = 0.61\,\mathrm{cm}, \qquad h_2 = 7.29\,\mathrm{cm} \qquad \Rightarrow v_n = -16.99\,\mathrm{cm/s}$$

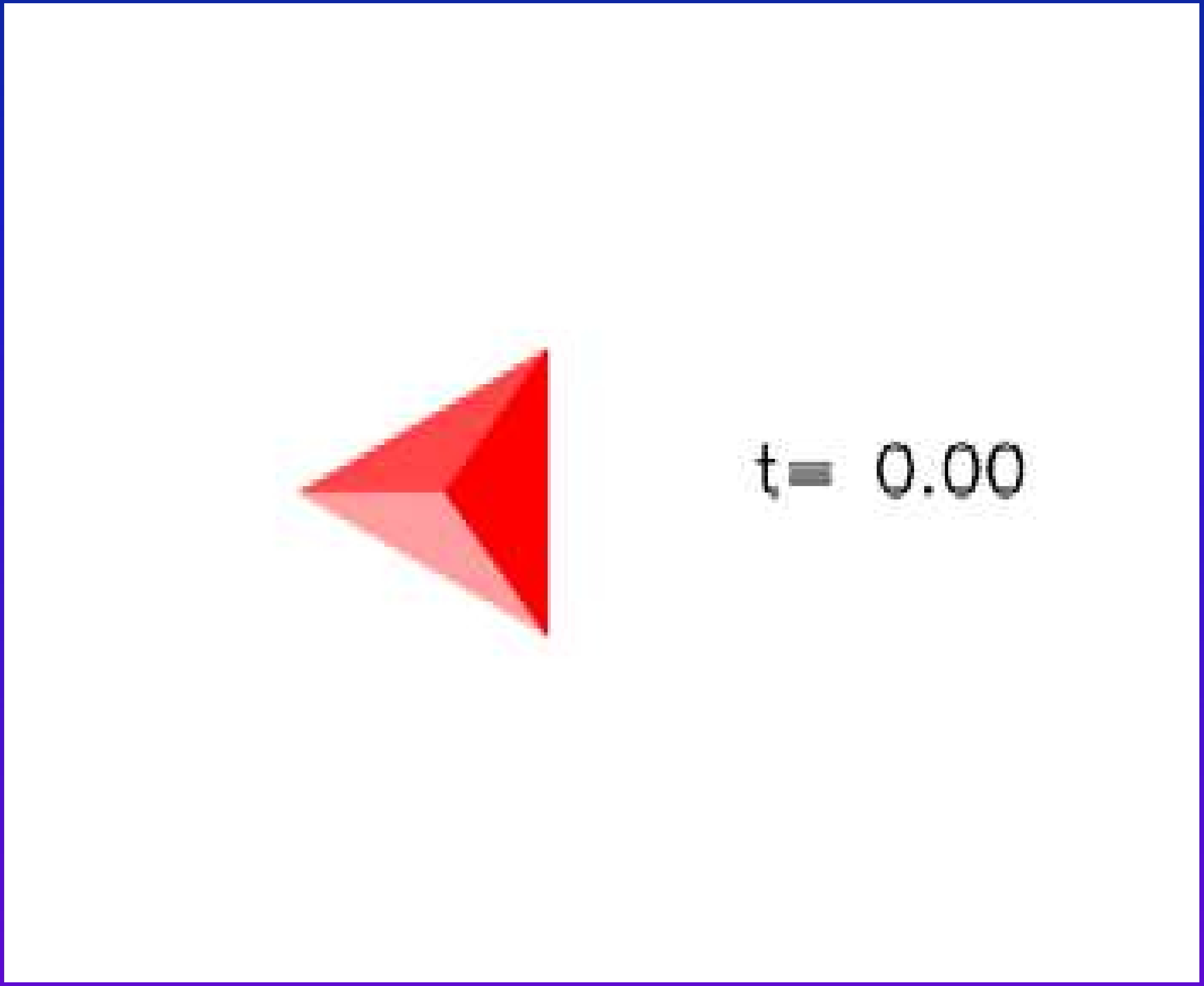
- lies within 10% of the measured value of $v_n = -15.4\,\mathrm{cm/s}$

Gray, Tai & Noelle (2003) *J. Fluid Mech.* **491**, 161-181.

Proposed defence for the Schneefernerhaus, Zugspitze

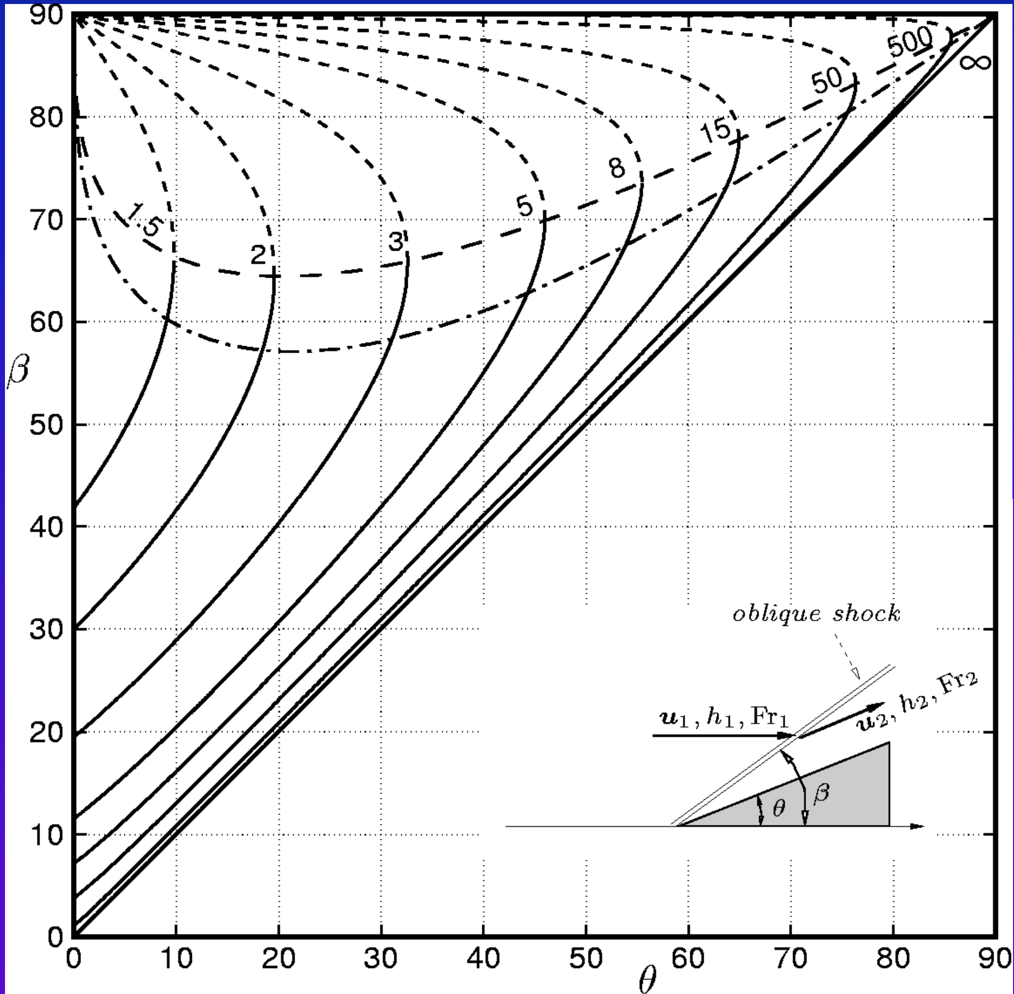


- Use avalanche model to compute the flow past obstacles



Gray, Tai & Noelle (2003) *J. Fluid Mech.* **491**, 161-181.

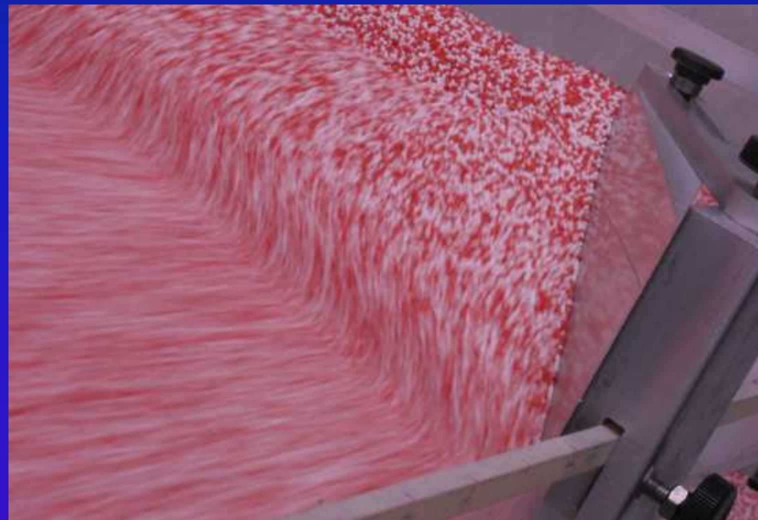
Weak, strong and detached oblique shocks



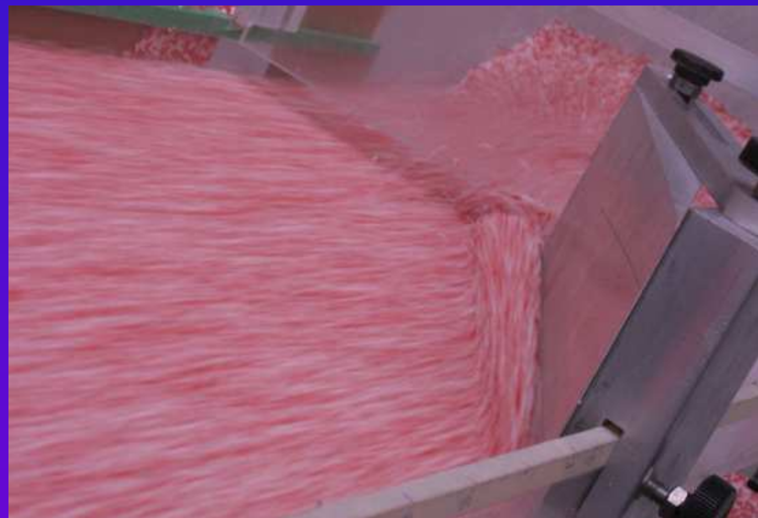
$$\tan\theta = \frac{\tan\beta\left(\sqrt{1+8\mathrm{Fr}_1^2\sin^2\beta}-3\right)}{2\tan^2\beta-1+\sqrt{1+8\mathrm{Fr}_1^2\sin^2\beta}}.$$

Gray & Cui (2007) *J. Fluid Mech.* **579**, 113-136.

$Fr_1 = 5$, $\theta = 20°$, $\zeta = 38°$

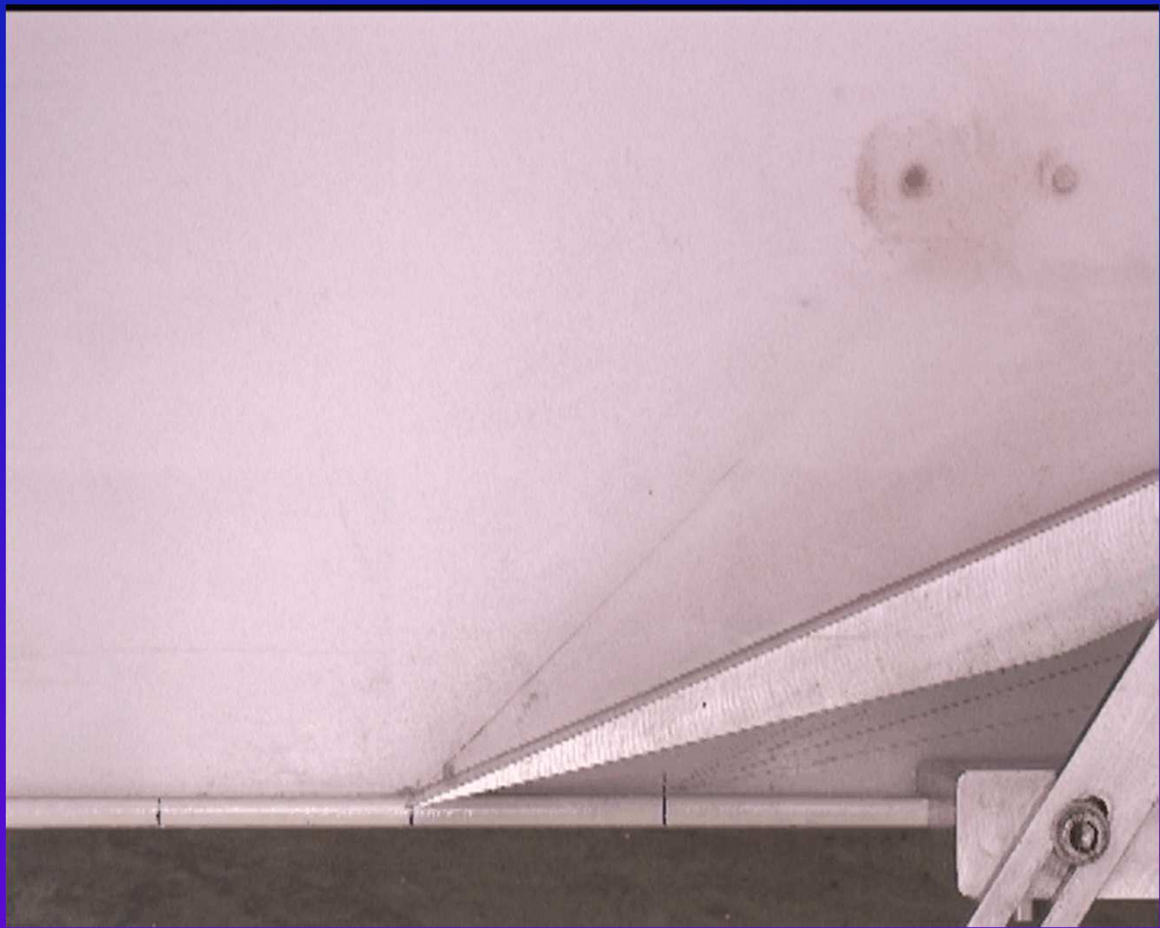


$\beta_s = 86.2°$ $(78° \pm 2°)$



$\beta_w = 30.7°$ $(29° \pm 1°)$

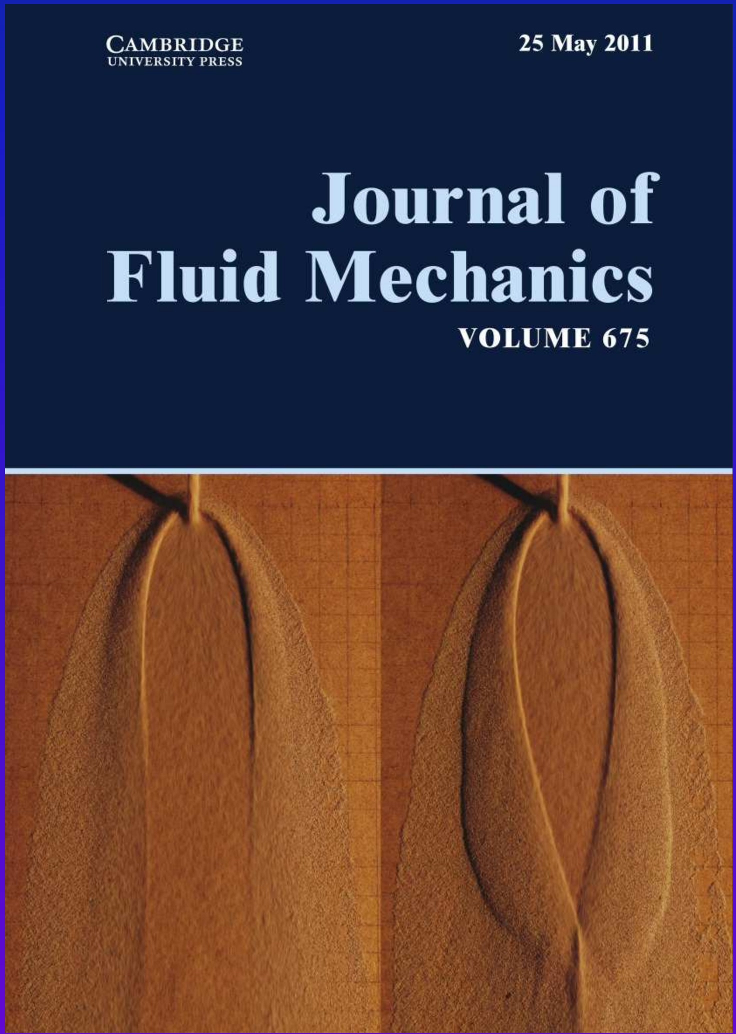
Weak Oblique Shock



Strong Oblique Shock

Detached Oblique Shock

Granular jets and hydraulic jumps on an inclined plane

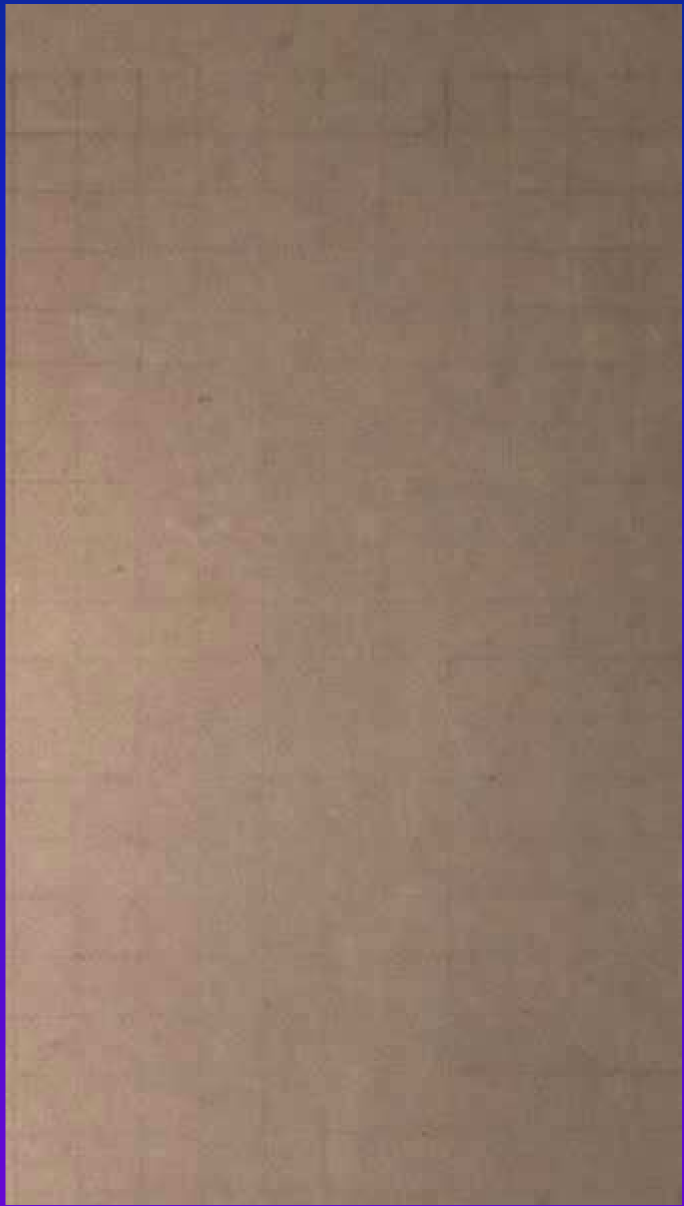


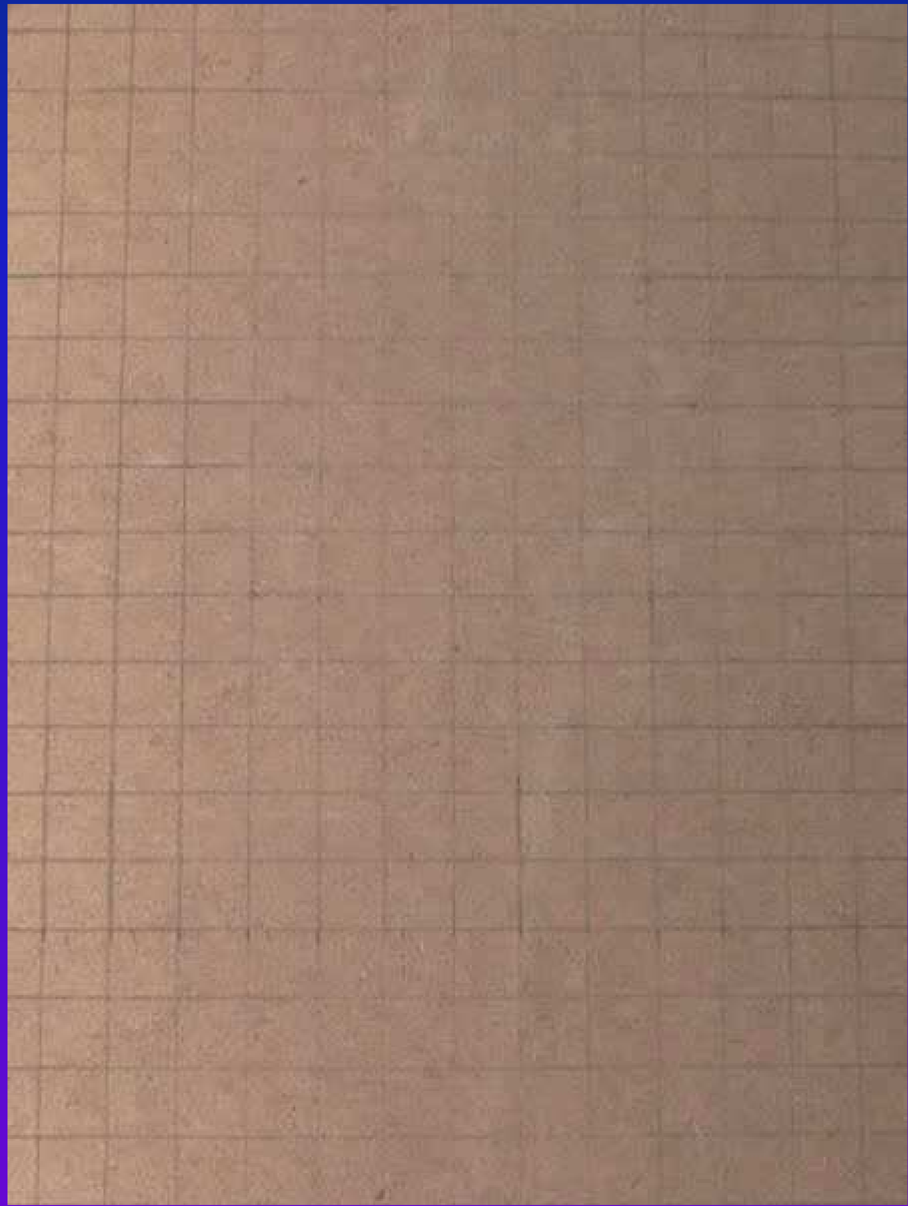
- Oblique impingement of an inviscid jet (Hasson & Peck 1964)
- Friction law for rough beds

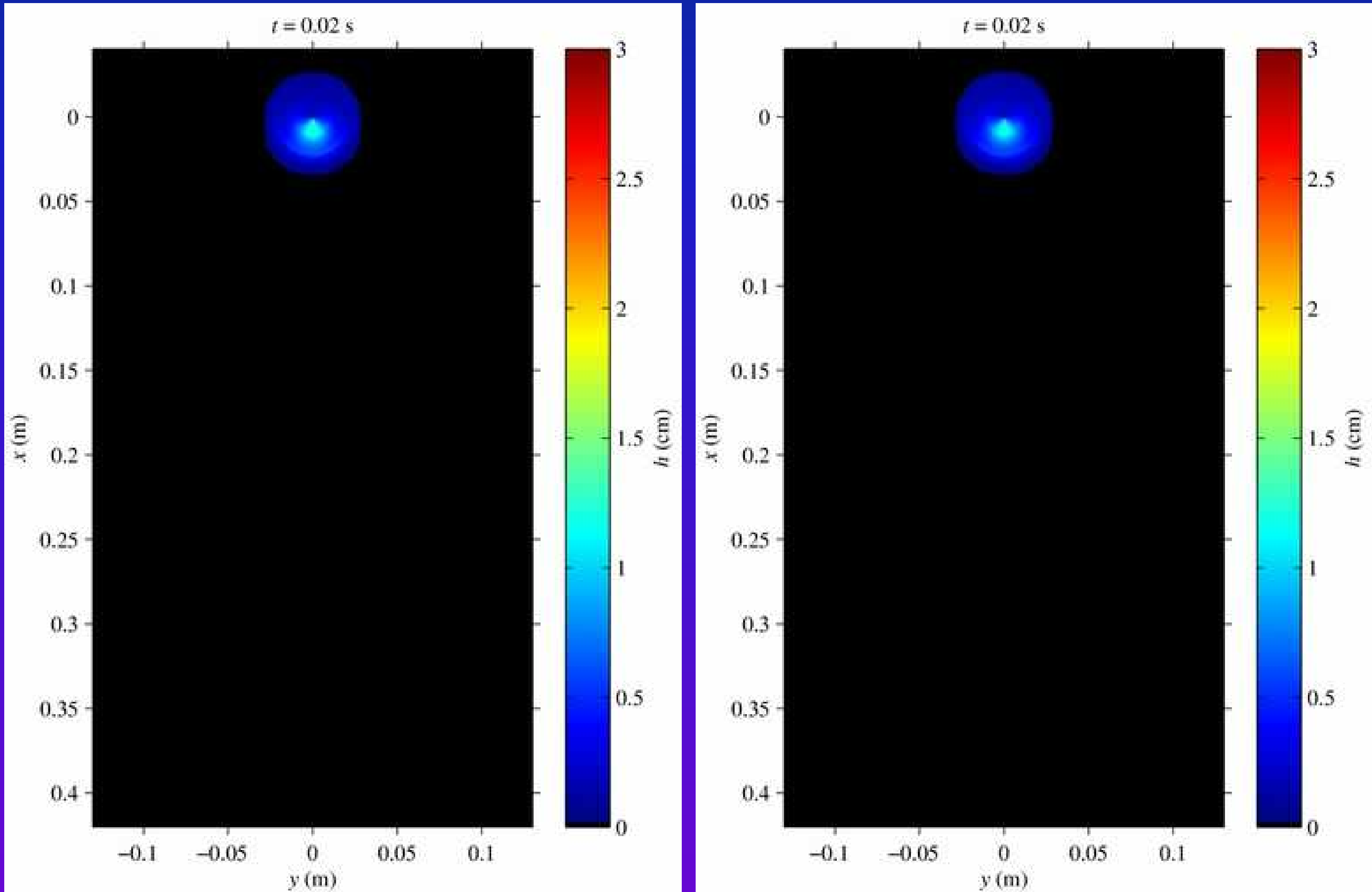
$$\mu = \tan\zeta_1 + \frac{\tan\zeta_2 - \tan\zeta_1}{1 + \beta h/(\mathcal{L}\mathrm{Fr})},$$

- including treatment of static material for $0 < \mathrm{Fr} < \beta$ (Pouliquen & Forterre 2002)

Johnson & Gray (2011) *J. Fluid Mech.* **675**, 87-116

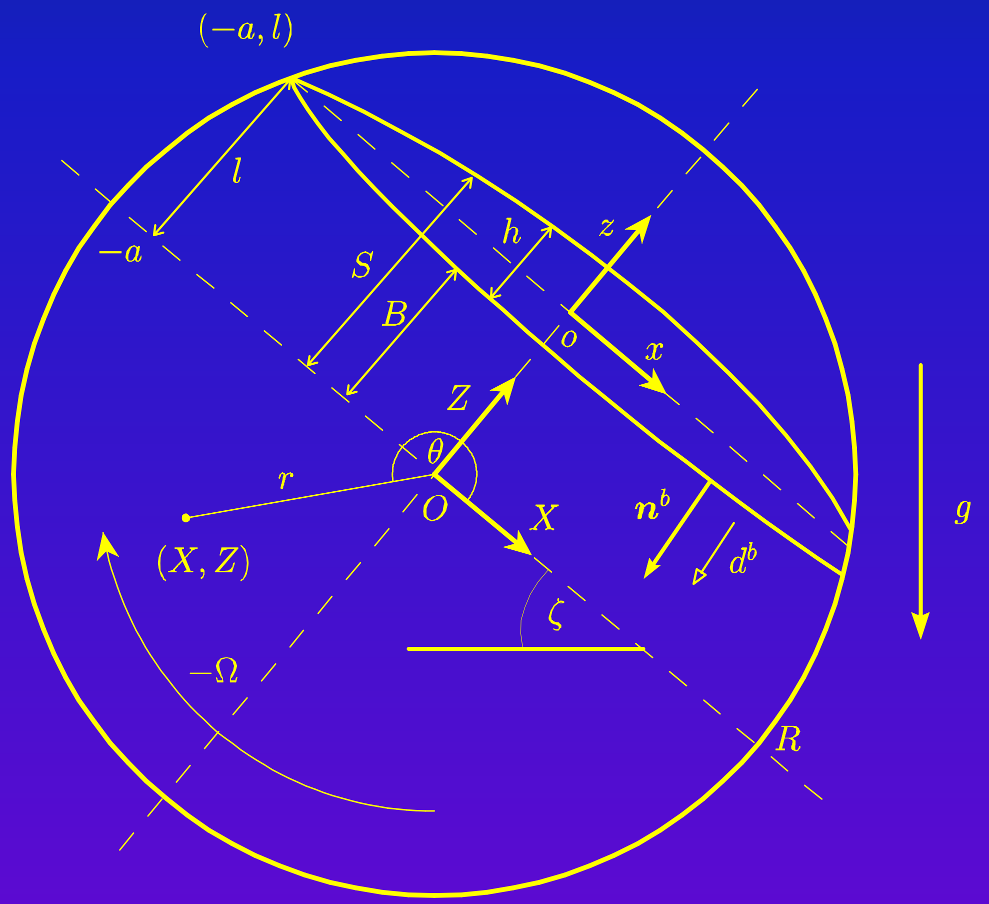





t = 0.02 s
x (m)
y (m)
h (cm)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
−0.1
−0.05
0
0.05
0.1
0
0.5
1
1.5
2
2.5
3
t = 0.02 s
x (m)
y (m)
h (cm)

Gray & Hutter (1997) *Contin. Mech. Thermodyn.* **9**(6), 341-345

Fluid and solid-like regions



Coupled avalanche model for flow in a rotating drum

- **Avalanche:** use hydraulic model with mass transfer

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) = -d^{b+},$$

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}\left(h\bar{u}^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\beta h^2\right) = hS - h\alpha\frac{\partial b}{\partial x} - \bar{u}d^{b+},$$

where S are source terms $\alpha = \cos\zeta$ and $\beta = K\alpha$

- **Interfacial conditions:** Coupling of fluid and solid regions by mass jump condition

$$[\![\rho(\boldsymbol{u}\cdot\boldsymbol{n}^b - v_n^b)]\!] = 0,$$

- **Solid rotating granular material:** treated as a rigid body rotating with angular velocity $\Omega(t)$

$$u^- = -\Omega Z, \quad w^- = \Omega X.$$

Steady-state solutions

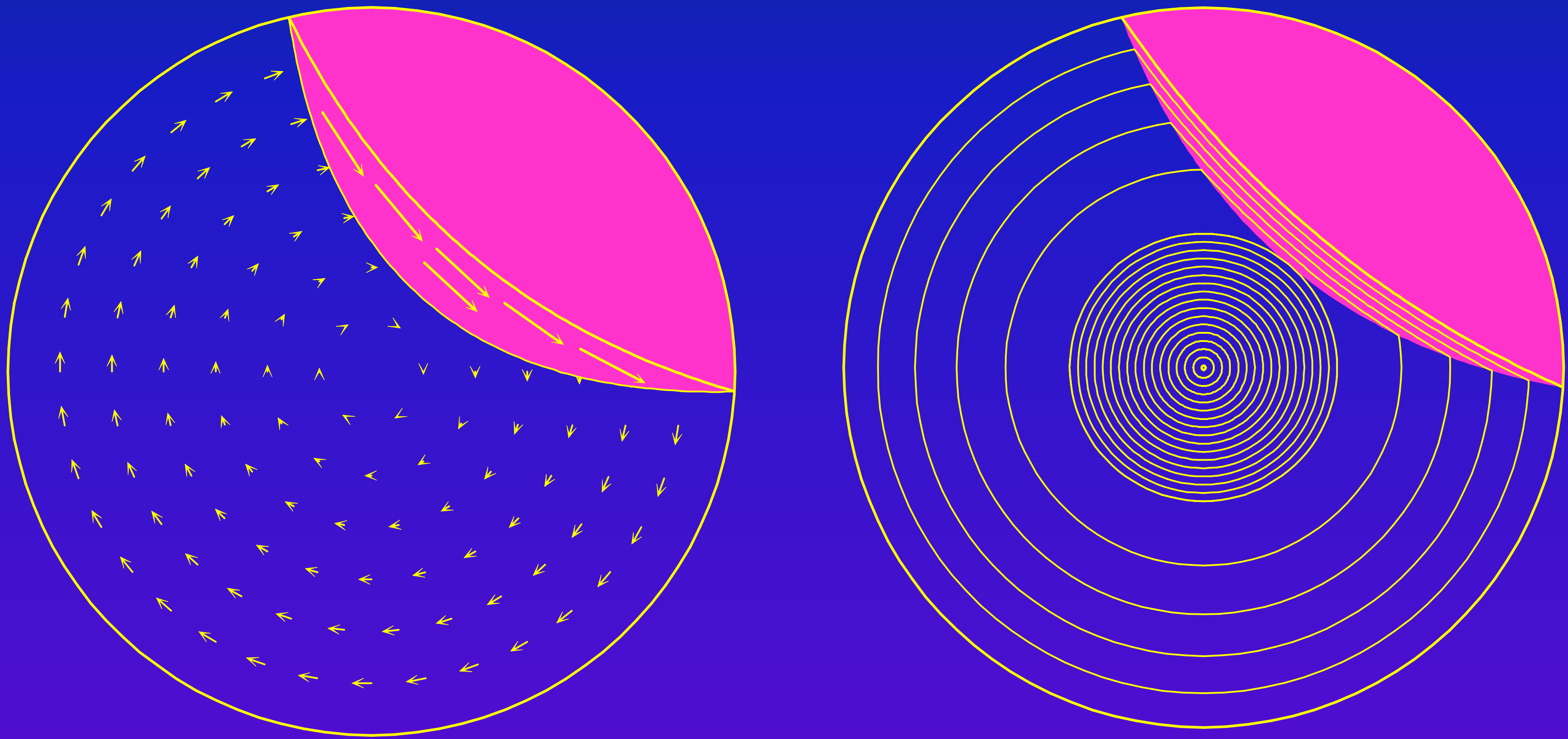
- constant angular velocity, Ω_0
- steady state $\partial/\partial t = 0$ and $v_n = 0$
- slope assumed to be non-accelerative, $\zeta = \delta$
- For classical smooth solutions

$$\begin{aligned} \frac{\partial}{\partial x}(h\bar{u}) &= (\rho^-/\rho^+)\left(\Omega_0 l \frac{\partial b}{\partial x} + \Omega_0 x\right), \\ h\bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}\beta h^2\right) &= -h\cos\zeta\frac{\partial b}{\partial x}. \end{aligned}$$

- We will investigate special class of solutions with

$$\bar{u} = \bar{u}_0,$$

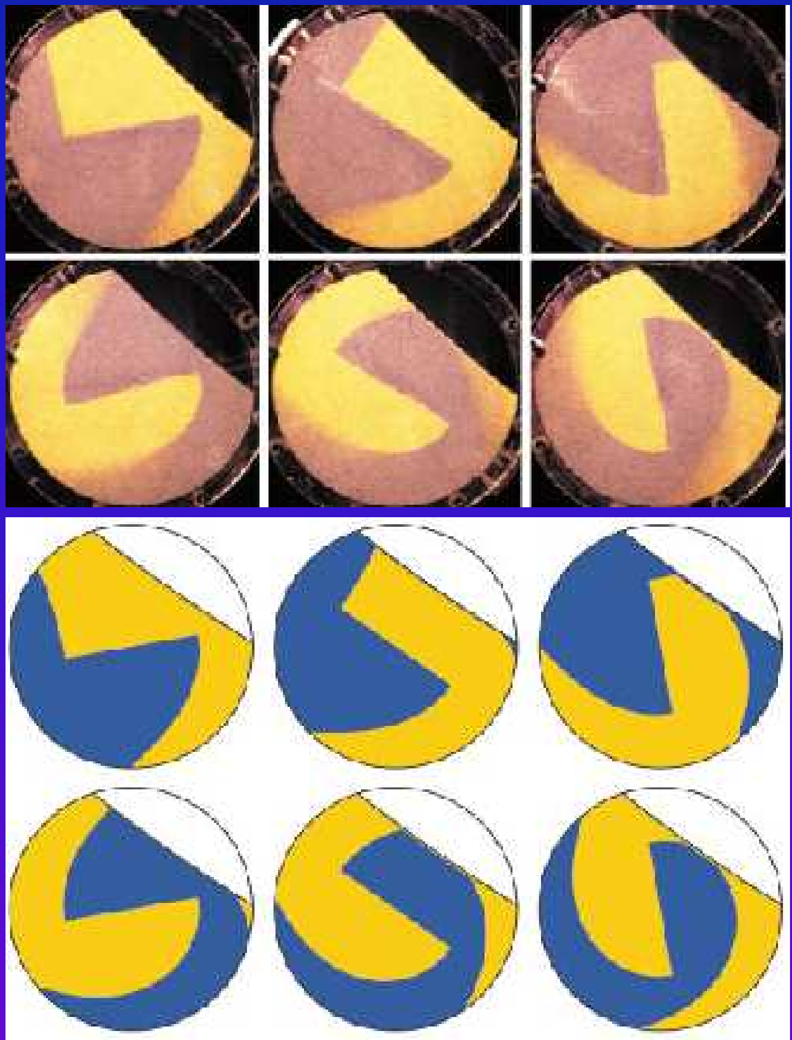
Steady-state drum solution and associated particle-paths

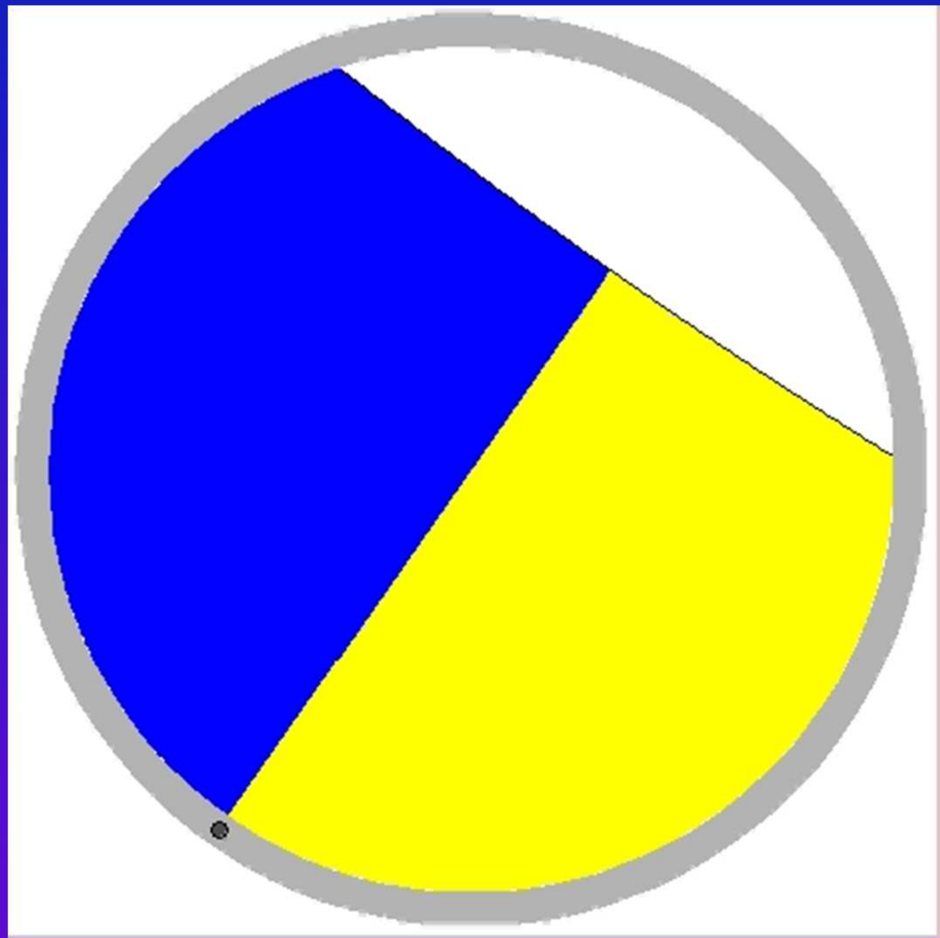


λ=3.00, l=0.60 B_0=0.37, l=0.60

- for fills greater than 50% fastest circuit times performed by grains close to drum wall
- for fills less than 50% the situation is reversed

Mixing of mono-disperse grains in a circular drum





Gray (2001) *J. Fluid. Mech.* **441**, 1-29

Rheology: We define viscosity using friction

We form the visco-plastic law: (Jop et al 2006)

$$\tau_{ij} = \mu P$$

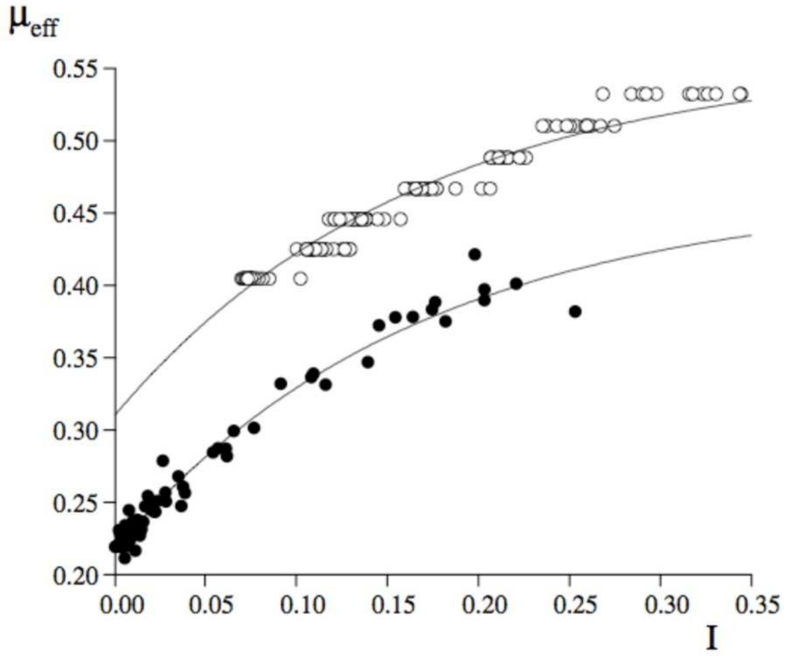
$$\tau_{ij} = \frac{\boldsymbol{\mu P}}{\mid \dot{\gamma} \mid}\dot{\gamma}_{ij}$$

using the phenomenological $\mu(I)$ rheology:

$$\mu(I) = \mu_s + \frac{\mu_d - \mu_s}{I_0/I + 1}$$

$$I = \frac{D \mid \dot{\gamma} \mid}{\sqrt{P/\rho}}$$

Valid for dense flows



Gdr MIDI 2004, Dacruz et al 2005, Jop et al 2006

Courtesy Lydie Staron

Solver: We apply the Open-source Gerris (Popinet 2003)
http://gfs.sourceforge.net
(incompressible Navier-Stokes equations using a VOF method) (Popinet 2003, 2009)

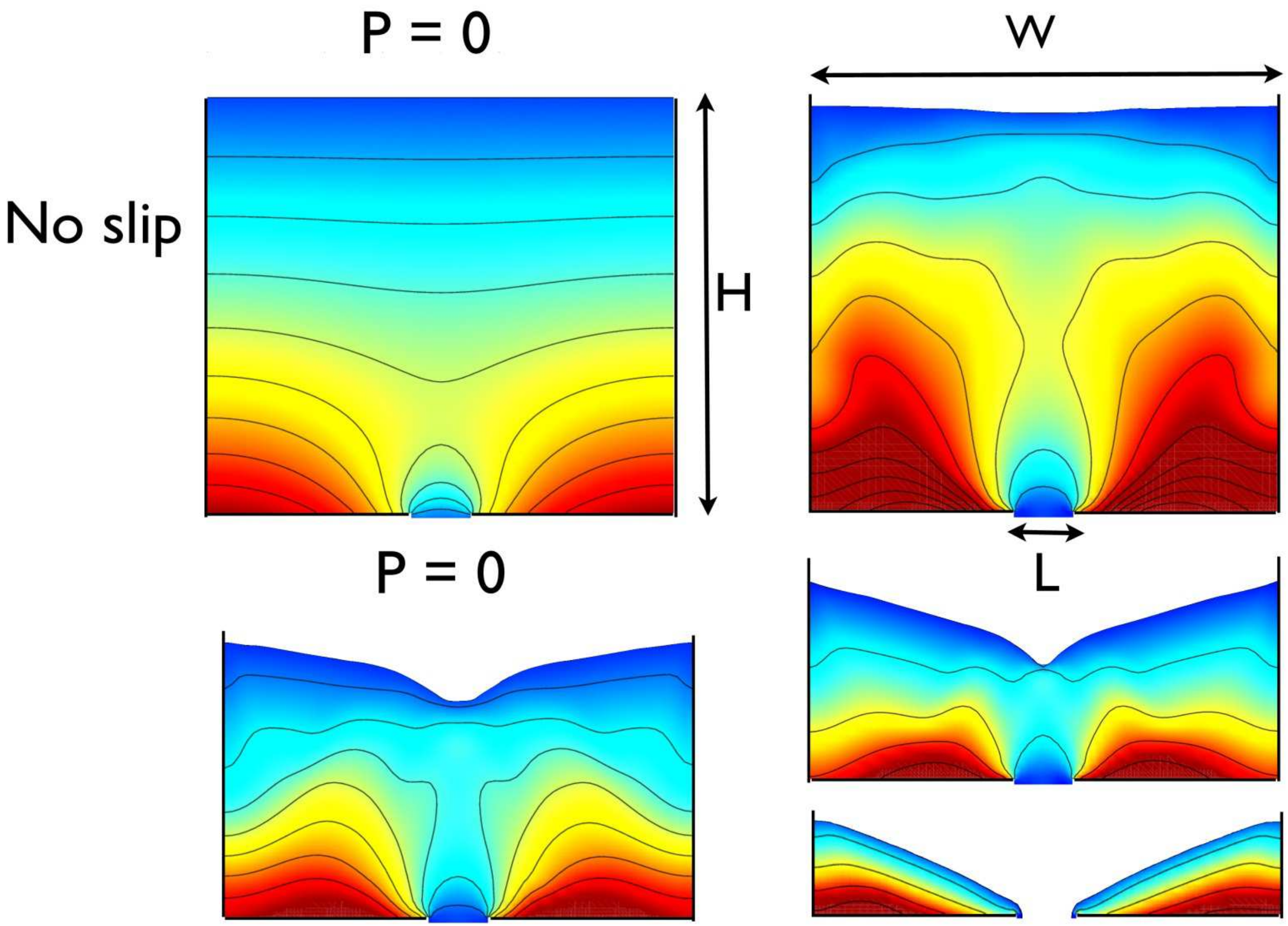
$$
\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{u} &= 0 \\
\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u}\right) &= -\boldsymbol{\nabla}p + \boldsymbol{\nabla}.(2\eta\boldsymbol{D}) + \rho\boldsymbol{g} \\
\frac{\partial c}{\partial t} + \boldsymbol{\nabla}.(c\boldsymbol{u}) &= 0 \\
\rho = c\,\rho_{\text{air}} &+ (1-c)\,\rho_{\text{grains}} \\
\eta = c\,\eta_{\text{air}} &+ (1-c)\,\eta_{\text{grains}}
\end{aligned}
$$

$\Rightarrow$ We chose $\rho_{\text{air}} << \rho_{\text{grains}}$
$\Rightarrow$ The free surface is solved in the course of time
$\Rightarrow$ We implement the viscosity:

$$\eta_{\text{grains}} = \min\left(\frac{\mu P}{\mid \dot{\gamma} \mid}, \eta_{max}\right),$$

Lagree, Staron & Popinet 2011

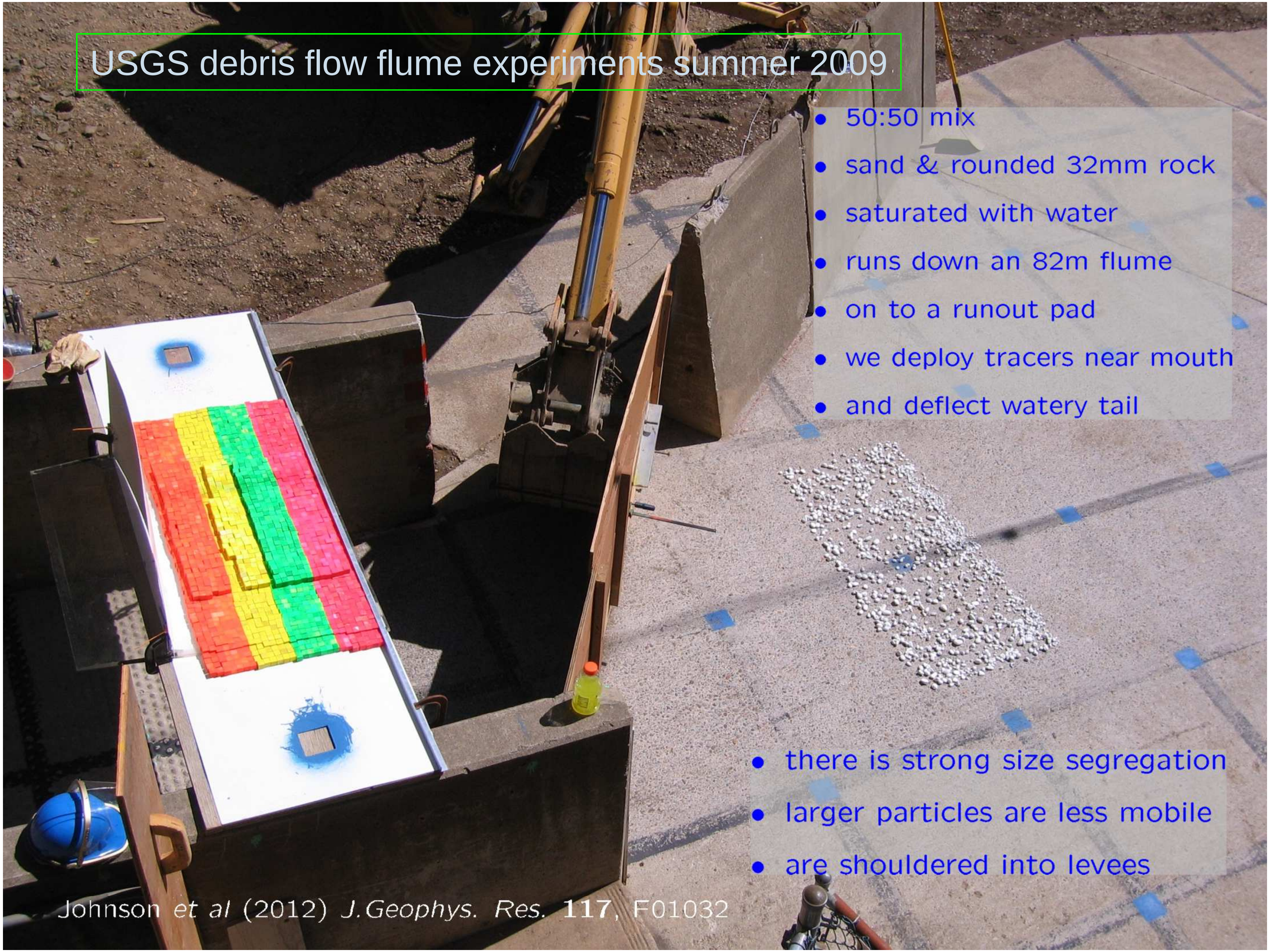
The Continuum Sand-Glass



We chose the following value for the rheological parameters:
$\mu_s = 0.32$, $\mu_d = 0.60$, $I_0 = 0.4$

Staron, Lagree & Popinet 2011

USGS debris flow flume experiments summer 2009

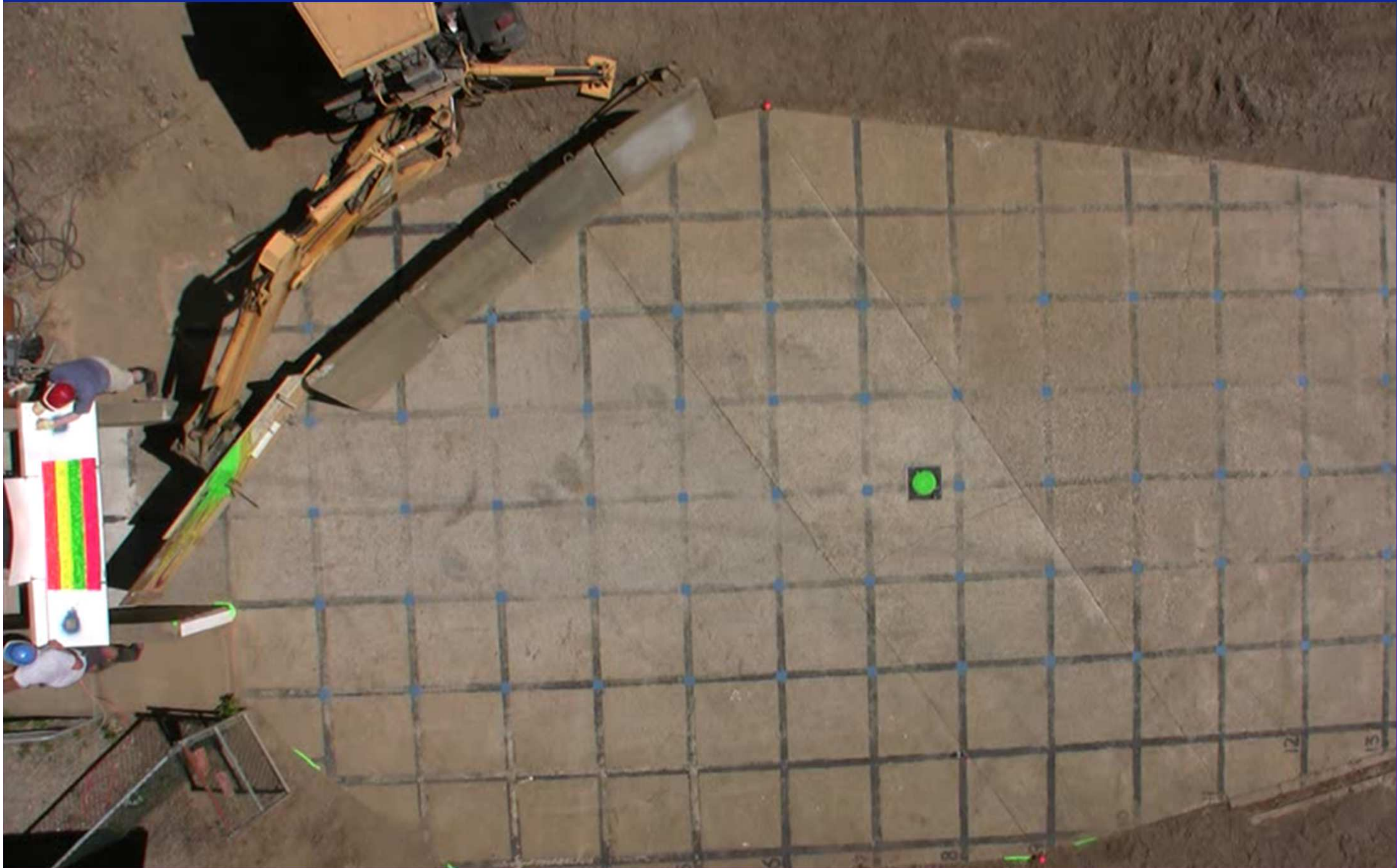


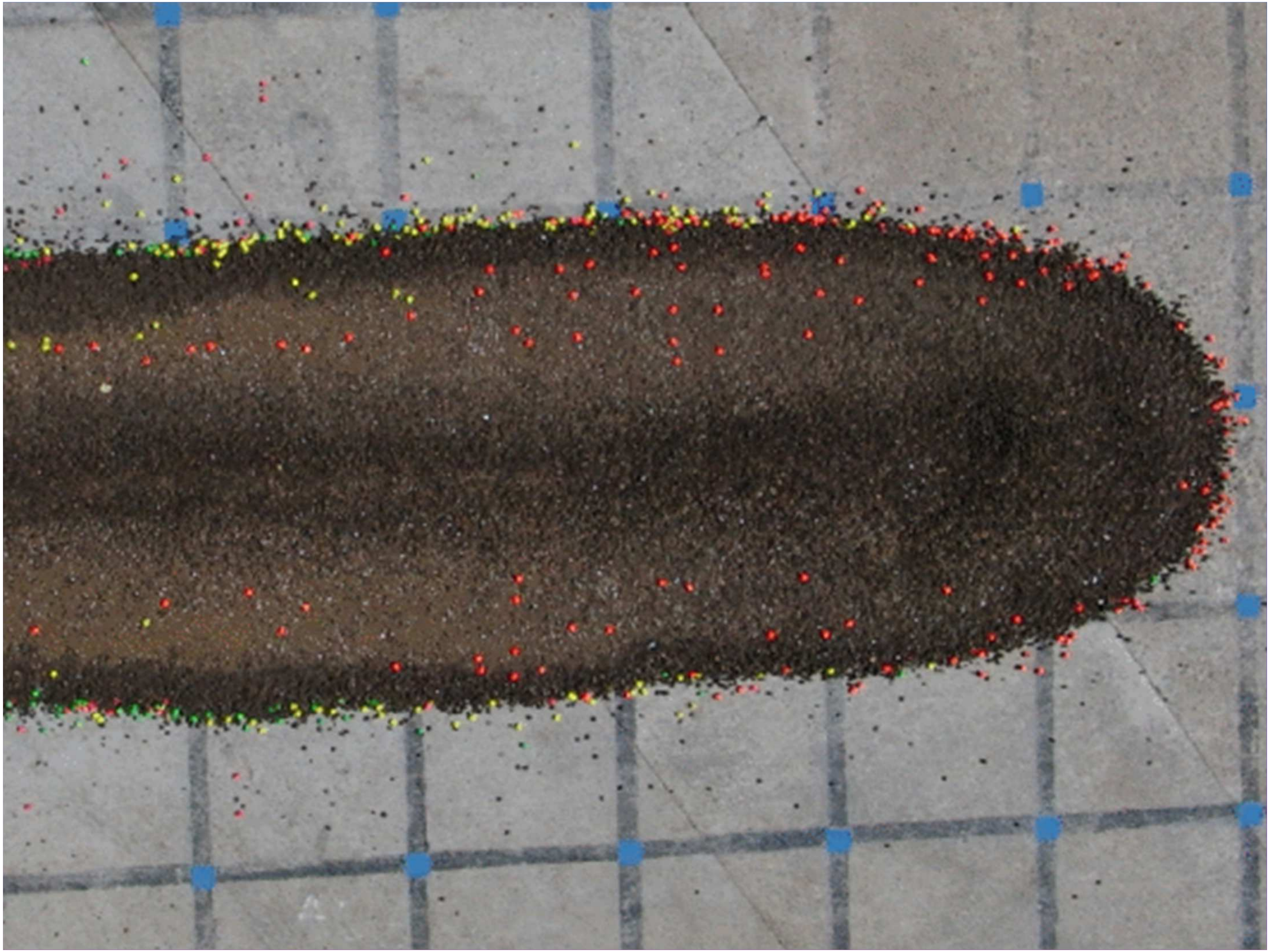
- 50:50 mix
- sand & rounded 32mm rock
- saturated with water
- runs down an 82m flume
- on to a runout pad
- we deploy tracers near mouth
- and deflect watery tail

- there is strong size segregation
- larger particles are less mobile
- are shouldered into levees

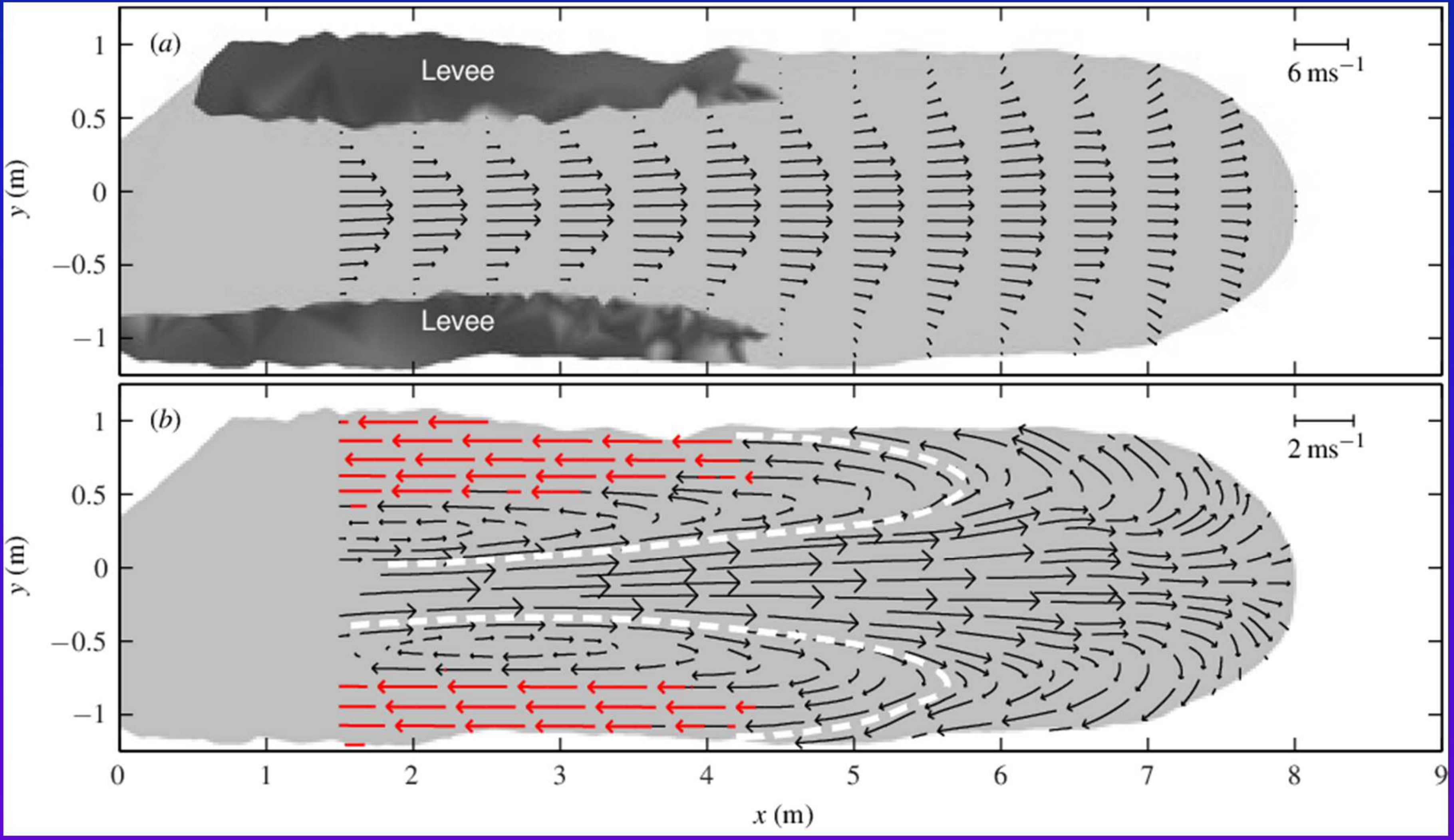
Johnson *et al* (2012) *J.Geophys. Res.* **117**, F01032

Measure velocity and levee emplacement time with 2cm cubes





Surface velocity in stationary and front centred frames



- $Oxyz$ are the downslope, cross-slope and normal directions

A simple kinematic model for 3D velocity field in the moving frame

- Bulk velocity $\mathbf{u} = (u, v, w)$ is assumed to be incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Integrating through the avalanche depth h

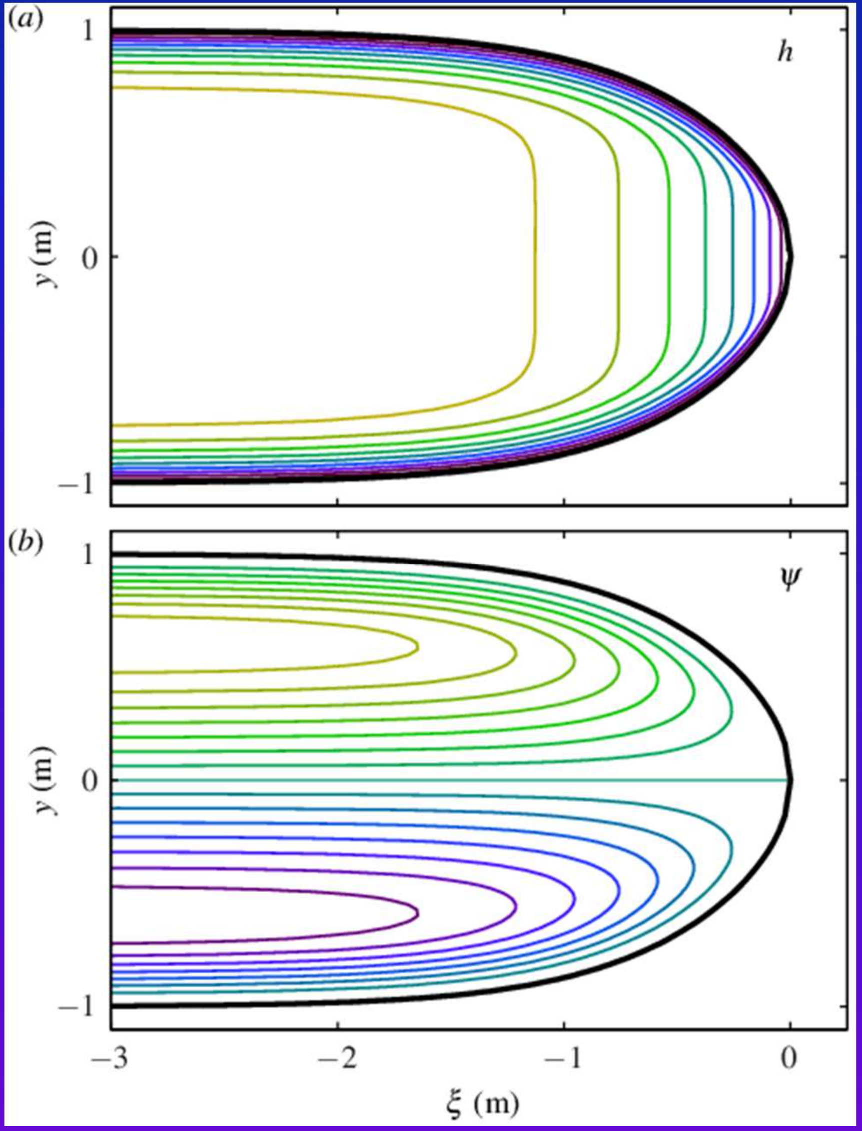
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0$$

where the depth-averaged velocity

$$\bar{u} = \frac{1}{h}\int_0^h u\,dz, \qquad \bar{v} = \frac{1}{h}\int_0^h v\,dz$$

- In frame $\xi = x - u_F t$ the bulk flow is steady

$$\frac{\partial}{\partial \xi}\left(h\left(\bar{u} - u_F\right)\right) + \frac{\partial}{\partial y}(h\bar{v}) = 0$$



- define a streamfunction

$$\frac{\partial \psi}{\partial y} = h(\bar{u} - u_F), \qquad \frac{\partial \psi}{\partial \xi} = -h\bar{v}$$

- empirical front shape

$$y_0(\xi) = W\sqrt{\tanh\left(-\frac{\xi}{W}\right)}$$

- self similar thickness h

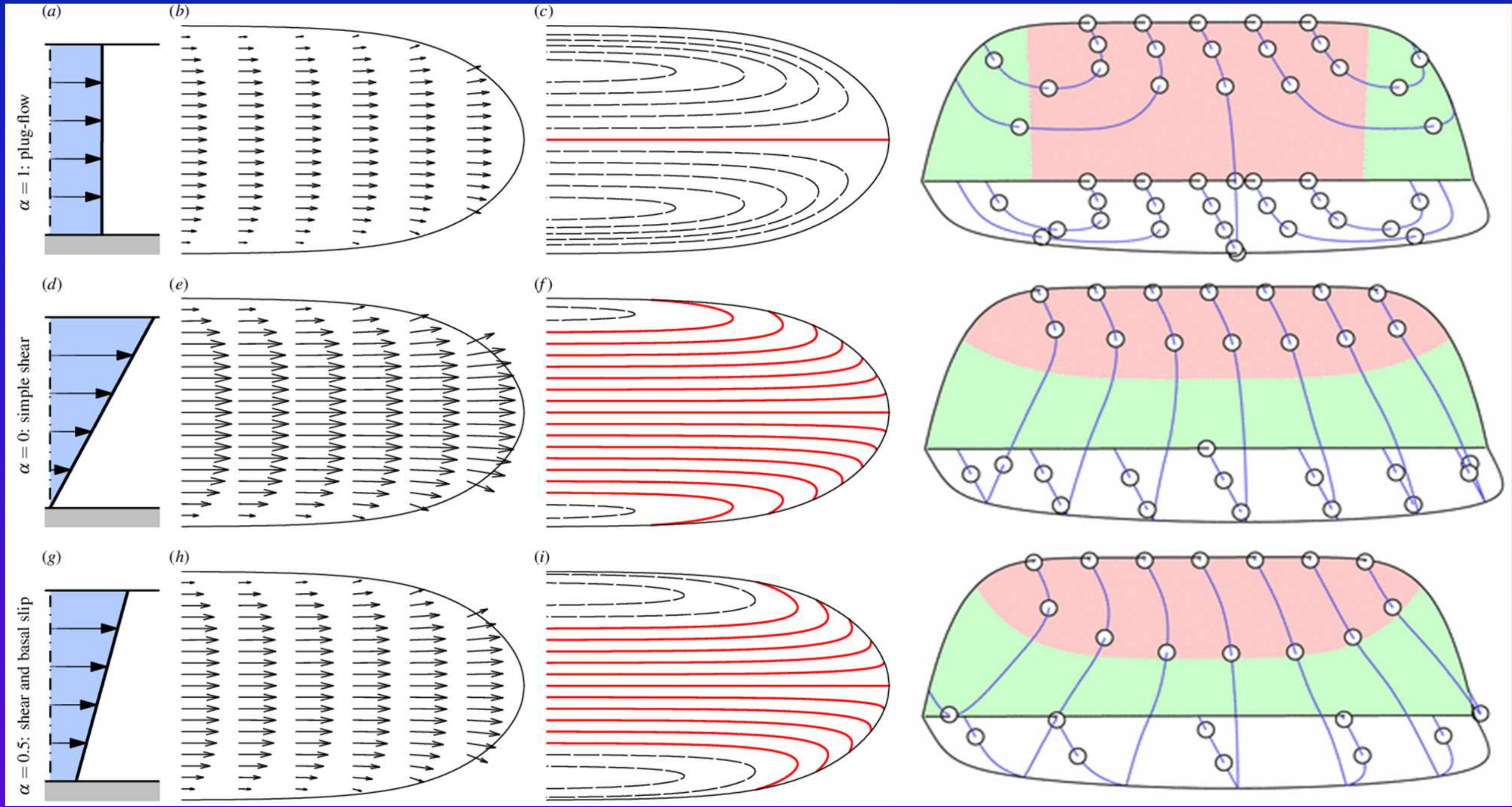
$$h(y_0, y) = \frac{H}{W}\left(\frac{y_0^{2n} - y^{2n}}{y_0^{2n-1}}\right)$$

- recirculating streamfunction

$$\psi(\xi, y) = \psi(y_0, y)$$

to approximate the flow

Reconstruction of the 3D velocity field



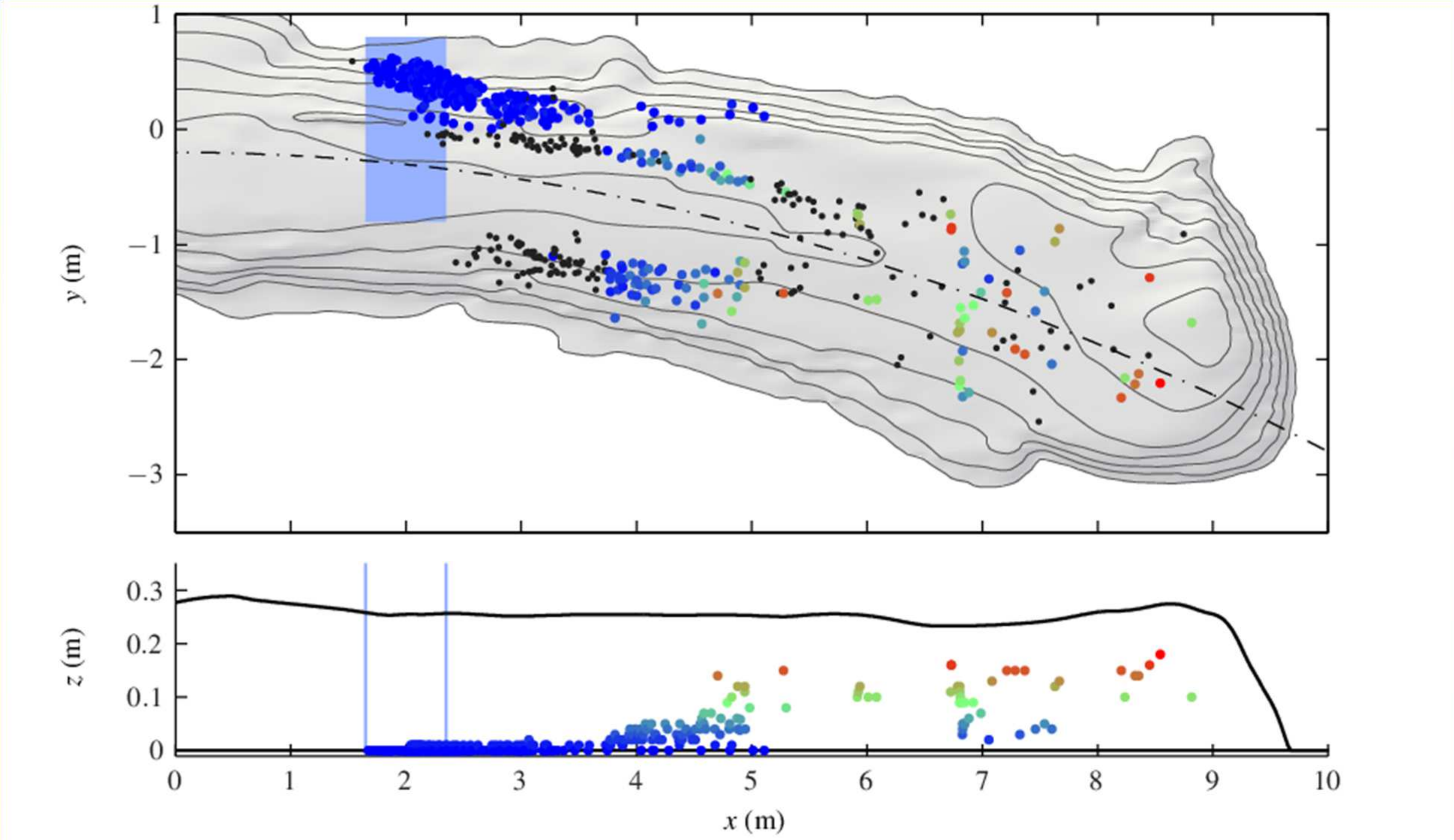
- assuming linear velocity profles with depth z

$$(u, v) = \left(\alpha + 2(1-\alpha)\frac{z}{h}\right)(\bar{u}, \bar{v})$$

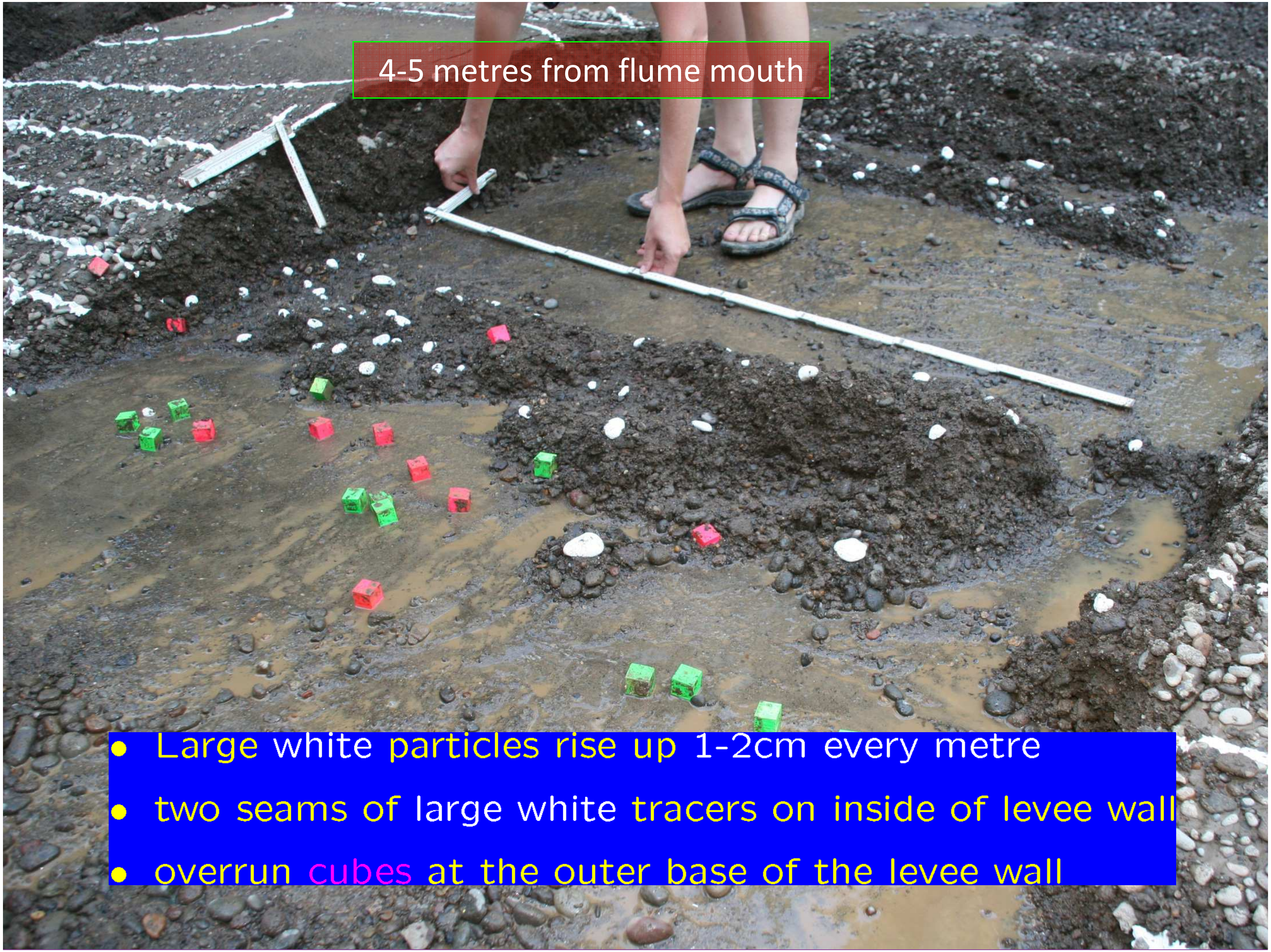
- half cubes lie at the surface
- mainly on top of the levee walls
- in reverse order, i.e. orange, yellow, green, pink

Formation of coarse grained lateral levees and finer grained interior

Large particle tracer stone heights

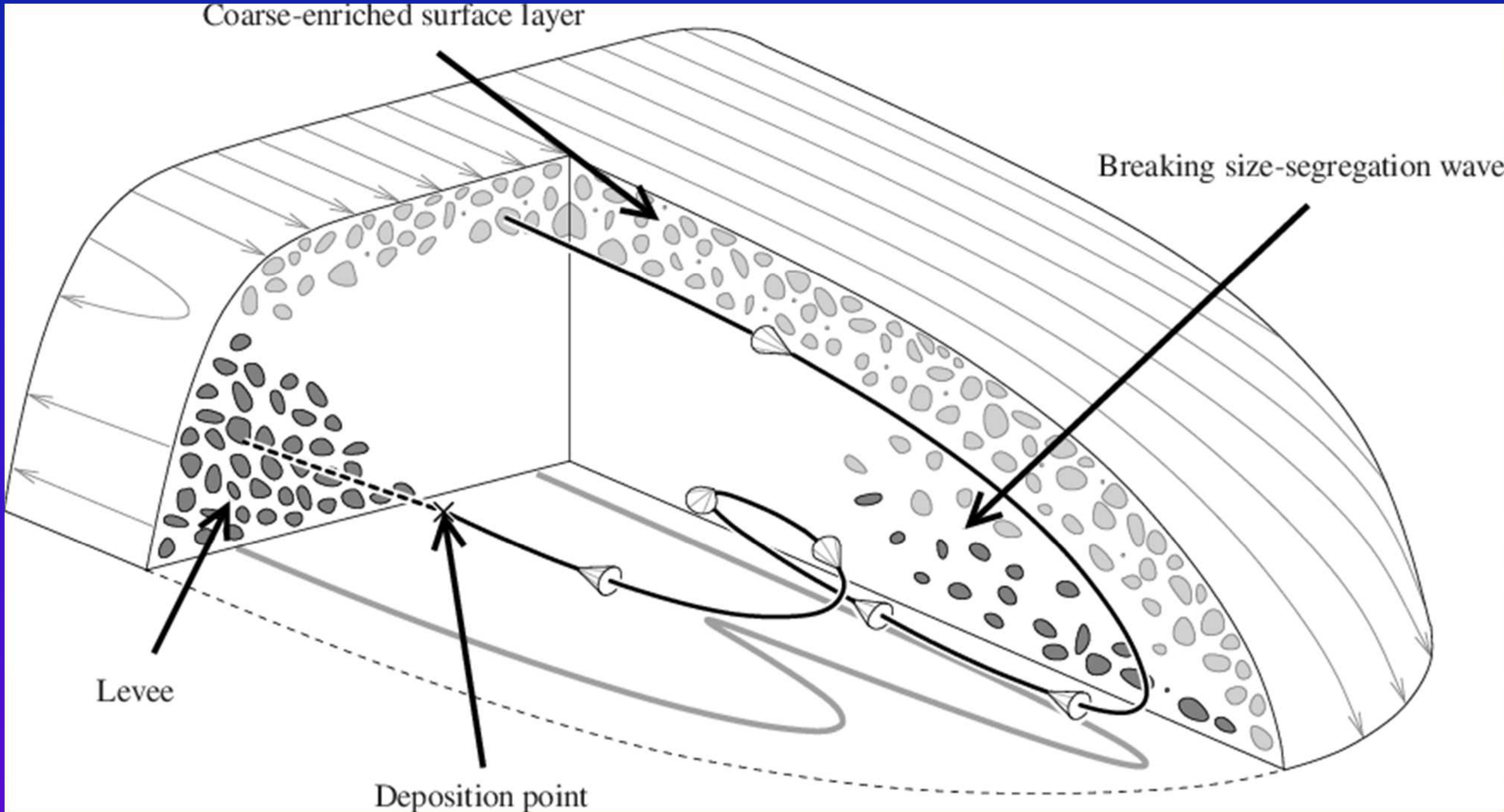


- Strong evidence for size segregation and recirculation
- BUT, stones never rise to the free surface again


4-5 metres from flume mouth
• Large white particles rise up 1-2cm every metre
• two seams of large white tracers on inside of levee wall
• overrun cubes at the outer base of the levee wall


At the flow front
• central white grains carried to flow front and reach 15cm height
• orange cubes overrun at the front

Schematic diagram of the levee formation process



- larger particles are shouldered to the sides to create levees
- this is an example of a segregation-mobility feedback effect