Particle size segregation in granular free-surface flows, by Nico Gray

MANCHESTER
1824

The University of Manchester
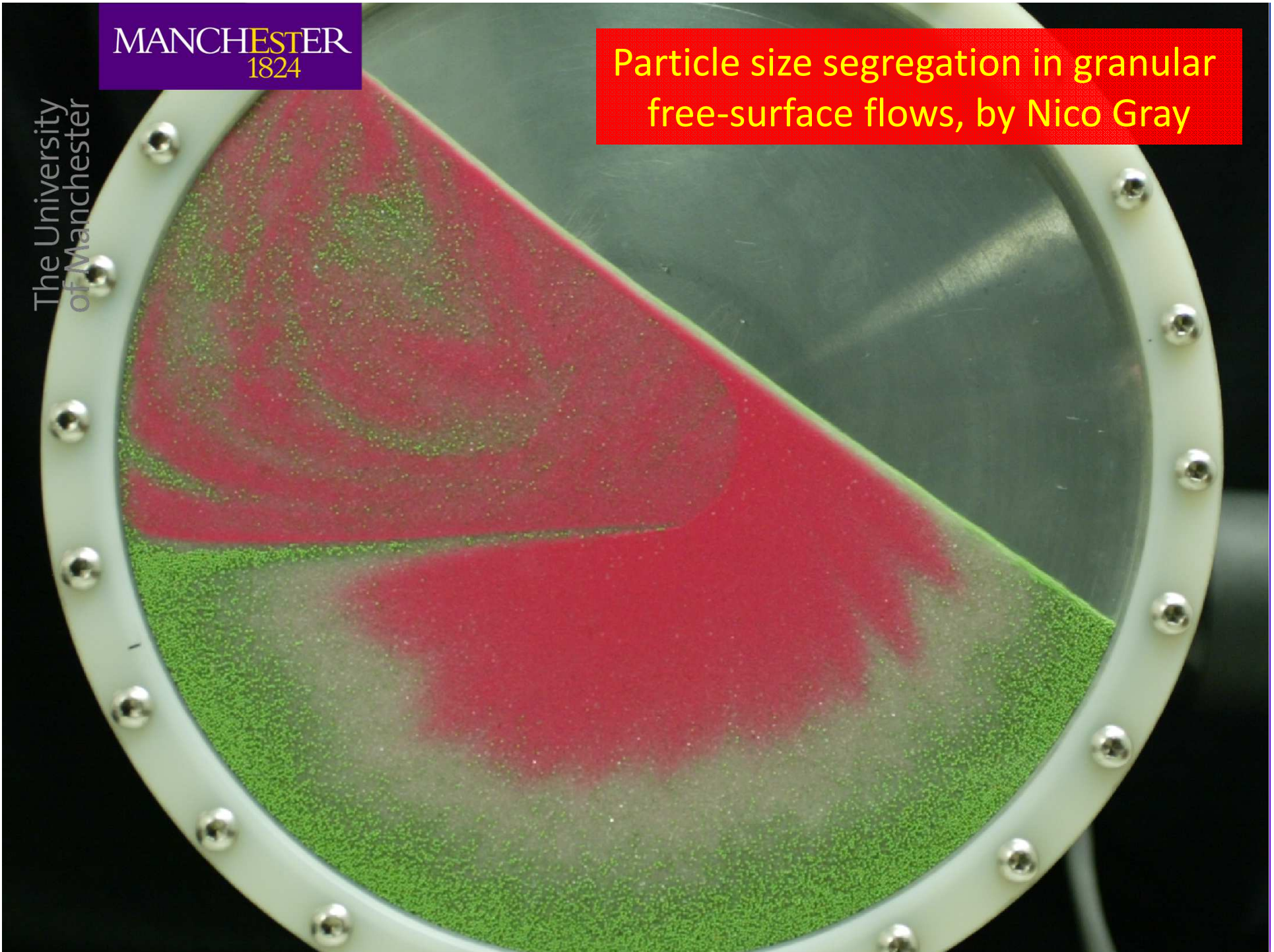
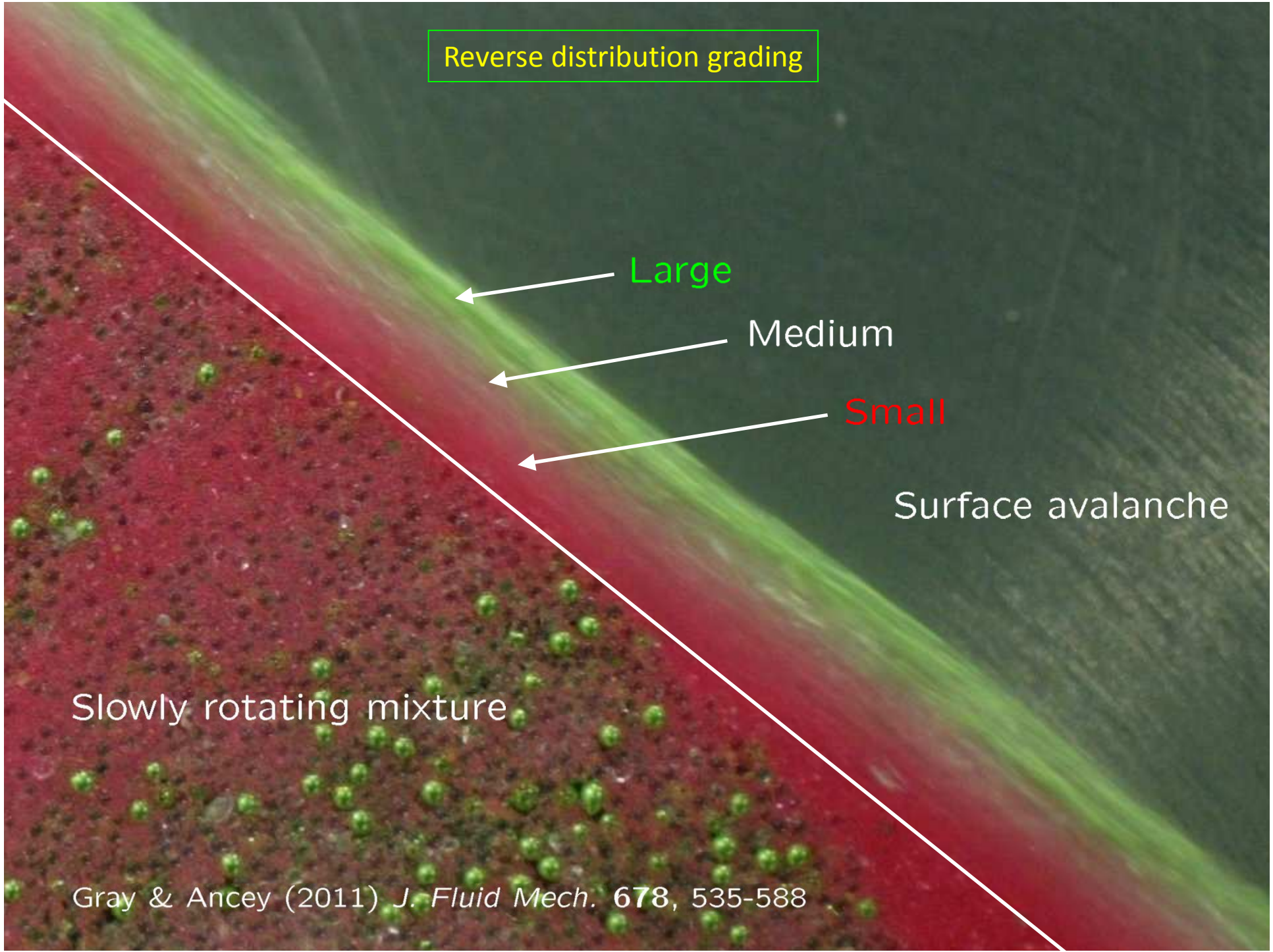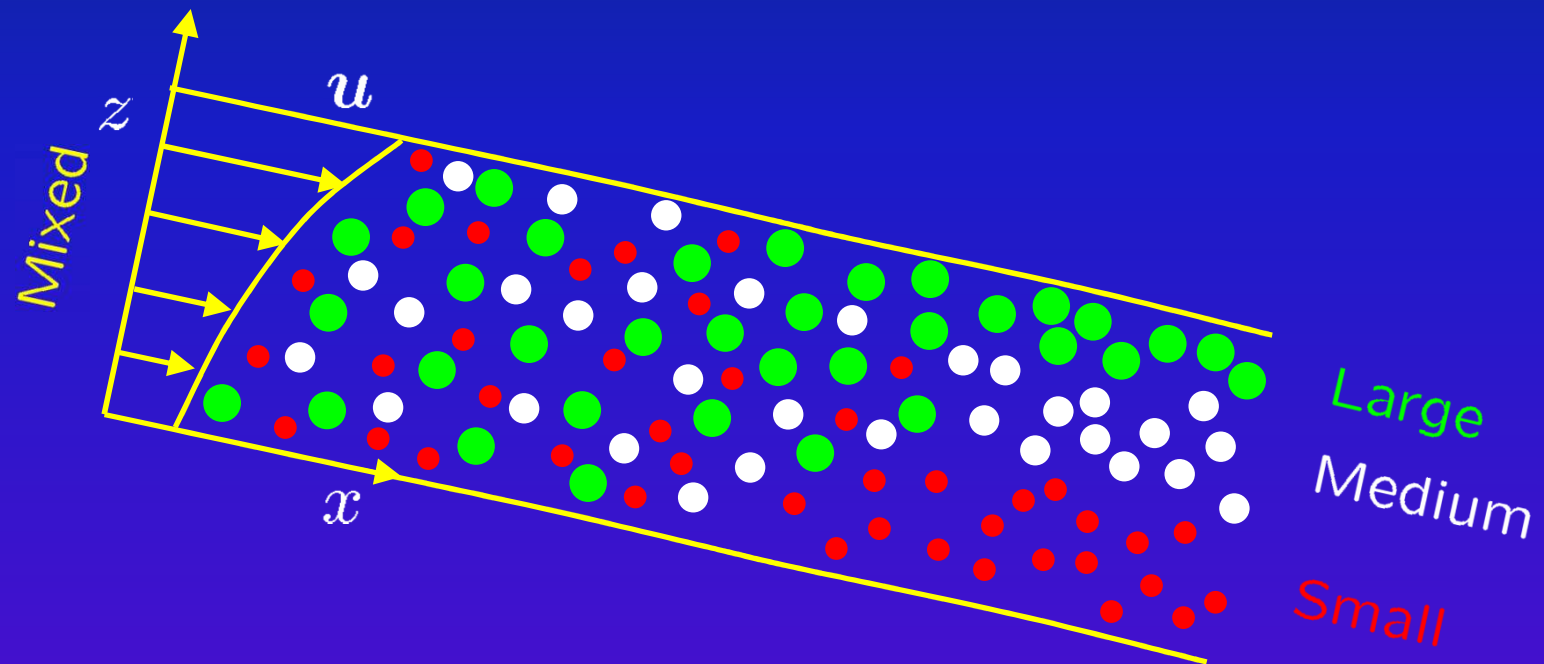Reverse distribution grading

Large

Medium

Small

Surface avalanche

Slowly rotating mixture

Gray & Ancey (2011) *J. Fluid Mech.* **678**, 535-588

Particle-size segregation and diffusive-remixing

- **kinetic sieving is the key segregation mechanism**

    – small particles more likely to fall down into gaps
    – and then force large particles up
    – to create inversely graded layers

- **diffusive-remixing competes against this**

3

- the volume fraction ϕ^ν of constituent ν, per unit volume of mixture, lies in the range

$$0 \le \phi^\nu \le 1.$$

and their sum

$$\sum_{\forall \nu} \phi^\nu = 1.$$

- In standard mixture theory the partial and intrinsic density, stress, pressure and velocity fields satisfy

$$\rho^\nu = \phi^\nu \rho^{\nu*}, \quad \boldsymbol{\sigma}^\nu = \phi^\nu \boldsymbol{\sigma}^{\nu*}, \quad p^\nu = \phi^\nu p^{\nu*}, \quad \boldsymbol{u}^\nu = \boldsymbol{u}^{\nu*}$$

- The bulk density and pressure are defined as

$$\rho = \sum_{\forall \nu} \rho^\nu, \quad p = \sum_{\forall \nu} p^\nu.$$

- Each constituent satisfies individual mass

$$\frac{\partial \rho^\nu}{\partial t} + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu) = 0,$$

and momentum balances

$$\frac{\partial}{\partial t}(\rho^\nu \boldsymbol{u}^\nu) + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu \otimes \boldsymbol{u}^\nu) = \nabla \cdot \boldsymbol{\sigma}^\nu + \rho^\nu \boldsymbol{g} + \boldsymbol{\beta}^\nu,$$

where $\otimes$ is the dyadic product and $\boldsymbol{g}$ is the gravitational acceleration vector.

- The interaction force $\boldsymbol{\beta}^\nu$ is the force exerted on phase ν by all the other constituents. Their sum

$$\sum_{\forall \nu} \boldsymbol{\beta}^\nu = \boldsymbol{0}.$$

5

- Particles have the same intrinsic density $\rho^{\nu*} = \rho_0$

- Acceleration terms are negligible in the z direction.

- The normal momentum balances sum to

$$\frac{\partial p}{\partial z} = -\rho g \cos \zeta,$$

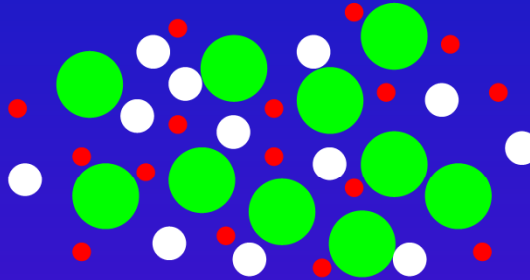 where g is the gravitational acceleration.

- Since ρ is constant and the free-surface $z = s(x,t)$ is traction free, this can be integrated to show that the pressure is lithostatic

$$p = \rho g(s - z)\cos \zeta.$$

- Compatible with existing avalanche models

6

- As small particles percolate downwards, they support less of the overburden pressure $\Rightarrow$ large grains support more of the load.

- suggests a partial/intrinsic pressure relation that differs from standard mixture theory

$$p^\nu = f^\nu p,$$

where $f^\nu \neq \phi^\nu$ determines the proportion of the pressure carried by each constituent. Their sum

$$\sum_{\forall \nu} f^\nu = 1.$$

7

- Interaction drag consists of three terms

$$\boldsymbol{\beta}^{\nu} = p\nabla f^{\nu} - \rho^{\nu} c (\boldsymbol{u}^{\nu} - \boldsymbol{u}) - \rho d \nabla \phi^{\nu},$$

c is the coefficient of inter-particle drag,
d is the coefficient of diffusive remixing

- barycentric (bulk) velocity defined by

$$\rho \boldsymbol{u} = \sum_{\forall \nu} \rho^{\nu} \boldsymbol{u}^{\nu}.$$

- The constituent velocities are assumed equal to the bulk velocity in the down- and cross-slope directions

$$u^{\nu} = u, \quad v^{\nu} = v.$$

8

- Assuming normal accelerations are negligible

$$\phi^\nu w^\nu = \phi^\nu w + (f^\nu - \phi^\nu)(g/c)\cos\zeta - (d/c)\frac{\partial \phi^\nu}{\partial z},$$

$f^\nu - \phi^\nu > 0$ particles rise
$f^\nu - \phi^\nu = 0$ no relative motion
$f^\nu - \phi^\nu < 0$ particles percolate downwards

- When any class of particles are in a pure phase they must carry all of the load

$$f^\nu = 1, \quad \text{when} \quad \phi^\nu = 1,$$

- When there are no particles of that phase, they cannot carry any of the load

$$f^\nu = 0, \quad \text{when} \quad \phi^\nu = 0.$$

9

- If any two constituents are found in isolation, the form of f^ν must reduce to bidisperse case, e.g.

$$f^l = \phi^l + B_{ls}\phi^l\phi^s,$$

- This suggests an additive decomposition

$$f^\nu = \phi^\nu + \sum_{\forall \mu} B_{\nu\mu}\phi^\nu\phi^\mu,$$

- $B_{(\nu\nu)} = 0, \quad \forall\nu$, no perturbations exerted by any constituent on itself

- $B_{\nu\mu} = -B_{\mu\nu}, \quad \forall\nu \neq \mu$, ν equal and opposite to μ

- then $\sum f^\nu = 1$ is automatically satisfied.

- Non-dimensionalizing on typical thickness H, length L and velocity U implies that

$$w^\nu = w + \sum_{\forall \mu} S_{\nu\mu} \phi^\mu - D_r \frac{\partial}{\partial z}(\ln \phi^\nu),$$

where

$$S_{\nu\mu} = \frac{Lg\cos\zeta}{HUc} B_{\nu\mu}, \quad D_r = \frac{DL}{H^2 U}.$$

- The non-dimensional segregation remixing equation for phase ν is therefore

$$\frac{\partial \phi^\nu}{\partial t} + \nabla \cdot (\phi^\nu \boldsymbol{u}) + \frac{\partial}{\partial z}\left(\sum_{\forall \mu} S_{\nu\mu} \phi^\nu \phi^\mu\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^\nu}{\partial z}\right),$$

- The multi-component theory yields two equations for the large and small particles

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{ls}\phi^l\phi^s) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right),$$

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s\phi^l) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).$$

- The summation condition $\sum \phi^\nu = 1$ implies
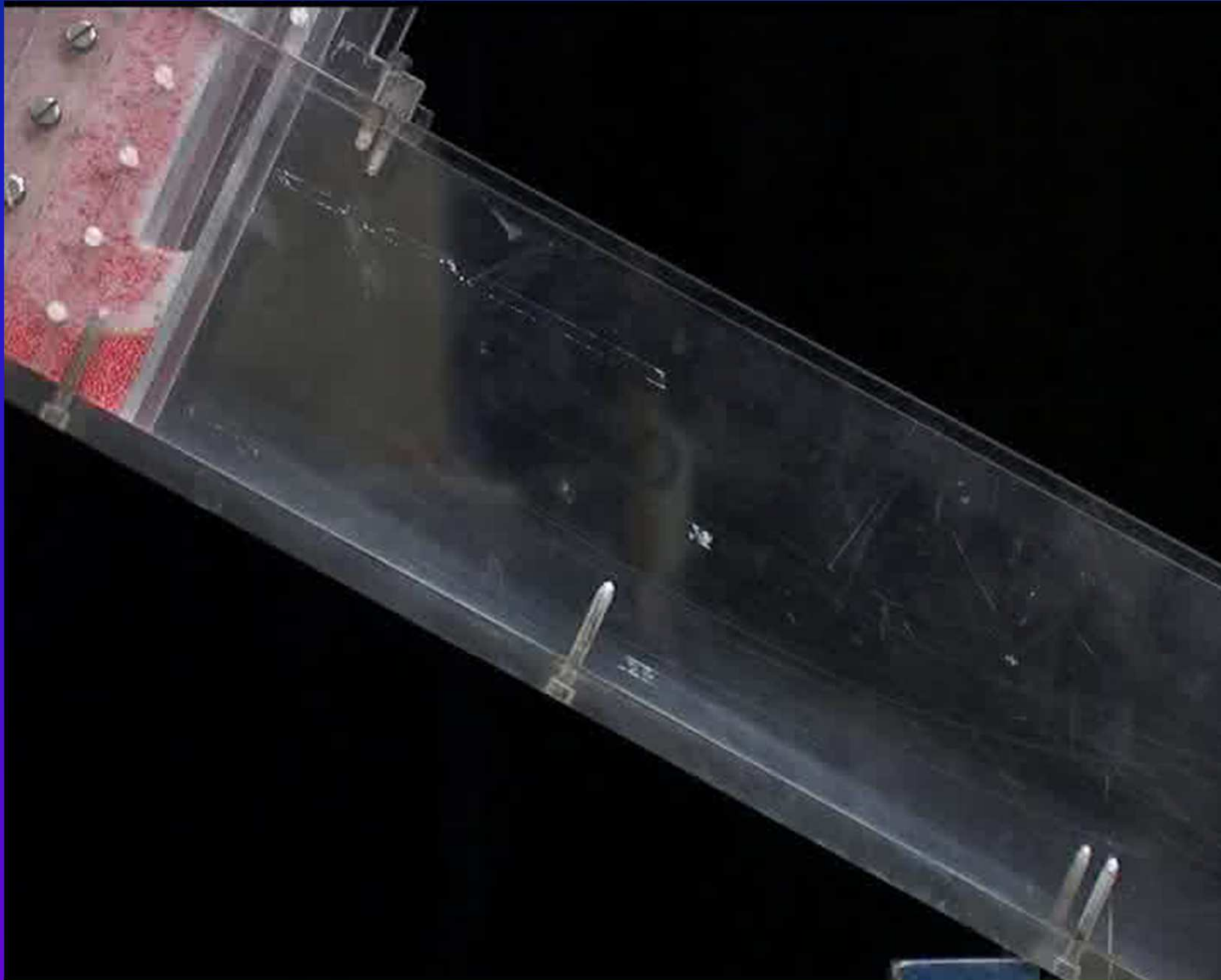
$$\phi^l + \phi^s = 1,$$

- Large particle concentration can be eliminated

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s(1-\phi^s)) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).$$
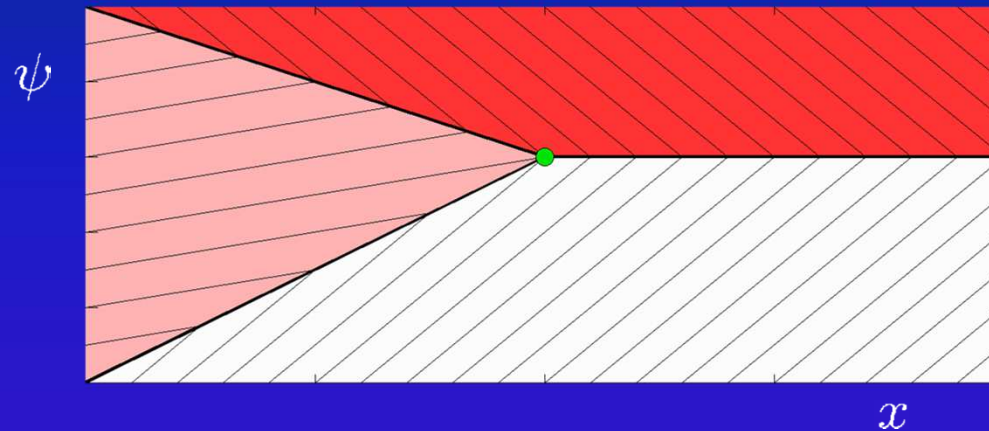
Gray & Thornton (2005) *Proc. Roy. Soc. A.* **461**, 1447-1473.
Gray & Chugunov (2006) *J. Fluid Mech.* **569**, 365-398.

Consistent with experiments of Savage & Lun (1988) and Vallance & Savage (2000)

Steady-state concentration shocks in absence of diffusive-remixing

- shock height $s(x)$ satisfies the jump condition

$$\left[\!\!\left[\phi u \frac{ds}{dx} + S_{ls}\phi(1-\phi) \right]\!\!\right] = 0 \quad \Rightarrow \quad u\frac{ds}{dx} = S_{ls}(\phi^+ + \phi^- - 1)$$
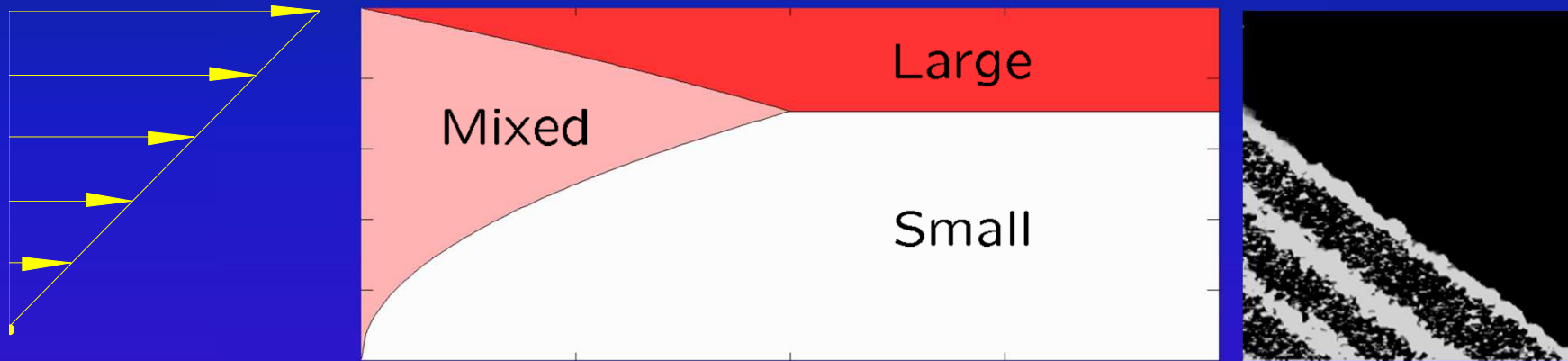
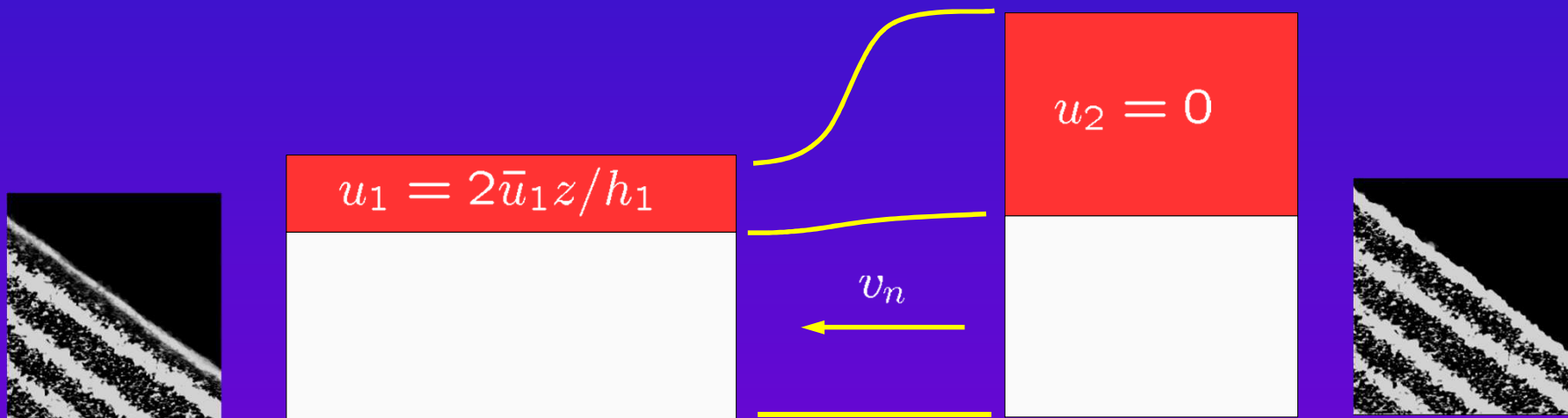- Using depth-integrated velocity coordinates

$$\psi = \int_0^z u(z')\,dz'$$

- this can be integrated to show there are three intersecting shocks for a homogeneous inflow with $\phi = \phi_0$

$$\psi_1 = S_{ls}\phi_0 x, \quad \psi_2 = 1 - S_{ls}(1-\phi_0)x, \quad \psi_3 = \phi_0$$

- within the flowing avalanche the particles "inverse-grade"

Mixed

Large

Small

- when they stop the upper layers expand more
- because there is a higher mass flux at the top

$u_2 = 0$

$u_1 = 2\bar{u}_1 z/h_1$

v_n

- Theory gives three equations

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{lm}\phi^l \phi^m + S_{ls}\phi^l \phi^s) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right)$$
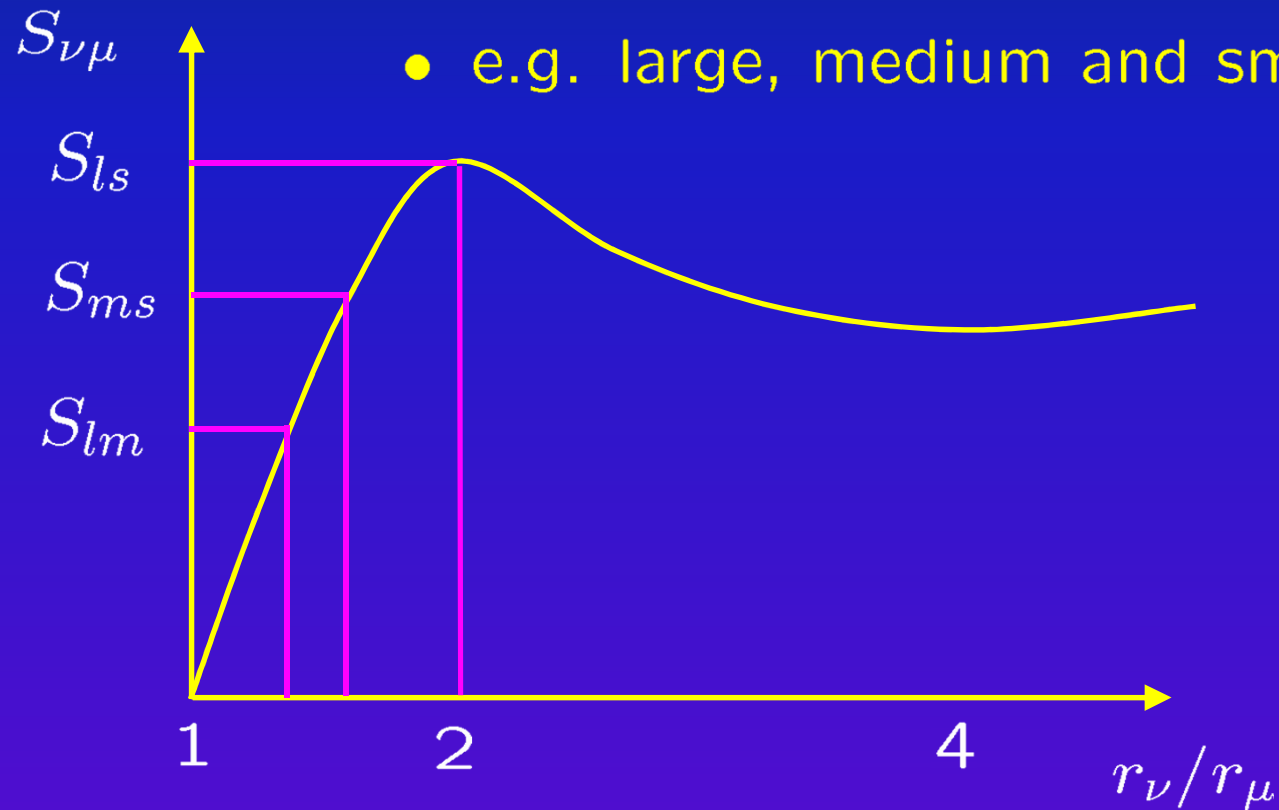
$$\frac{\partial \phi^m}{\partial t} + \nabla \cdot (\phi^m \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{lm}\phi^m \phi^l + S_{ms}\phi^m \phi^s) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^m}{\partial z}\right)$$

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{ls}\phi^s \phi^l - S_{ms}\phi^s \phi^m) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right)$$

- Sum $\phi^l + \phi^m + \phi^s = 1$ allows ϕ^m to be eliminated

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{lm}\phi^l(1 - \phi^l - \phi^s) + S_{ls}\phi^l \phi^s) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right)$$

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{ls}\phi^s \phi^l - S_{ms}\phi^s(1 - \phi^l - \phi^s)) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right)$$

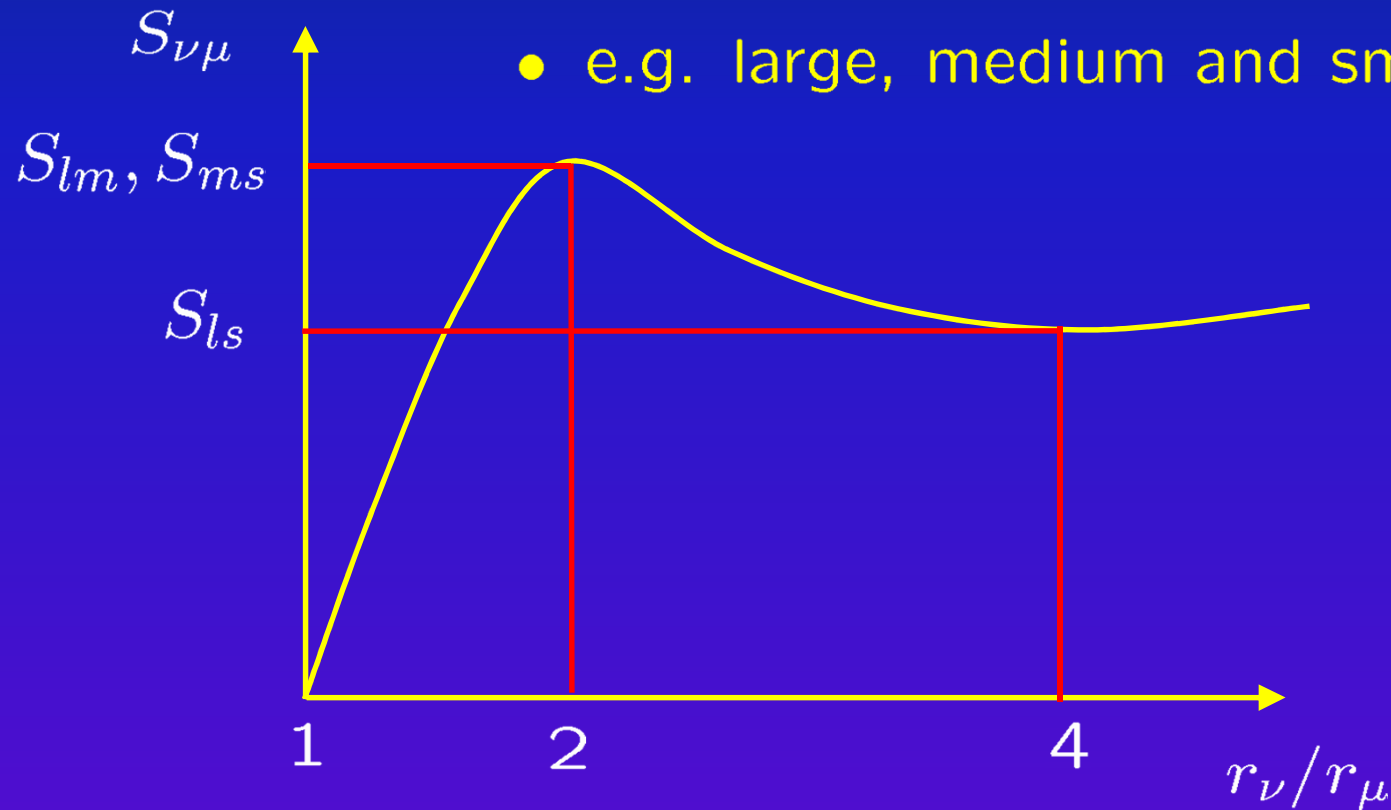- We will see diffusion is important to maintain well-posedness

16

- e.g. large, medium and small particles

- Golick & Daniels (2009) suggest segregation rate may have maximum at a grain size ratio of two

- $\Rightarrow$ segregation rates S_{ls}, S_{lm}, S_{ms} are not ordered

17

$S_{\nu\mu}$

S_{lm}, S_{ms}

S_{ls}

r_ν/r_μ

1 2 4

- e.g. large, medium and small particles

- Golick & Daniels (2009) suggest segregation rate may have maximum at a grain size ratio of two

- $\Rightarrow$ segregation rates S_{ls}, S_{lm}, S_{ms} are not ordered

18

- Defining $\phi = (\phi^l, \phi^m, \phi^s)^T$ system can be written as

$$\frac{D\phi}{Dt} + A\frac{\partial\phi}{\partial z} = 0,$$

$$A = \begin{pmatrix} S_{lm}\phi^m + S_{ls}\phi^s & S_{lm}\phi^l & S_{ls}\phi^l \\ -S_{lm}\phi^m & -S_{lm}\phi^l + S_{ms}\phi^s & S_{ms}\phi^m \\ -S_{ls}\phi^s & -S_{ms}\phi^s & -S_{ls}\phi^l - S_{ms}\phi^m \end{pmatrix}.$$

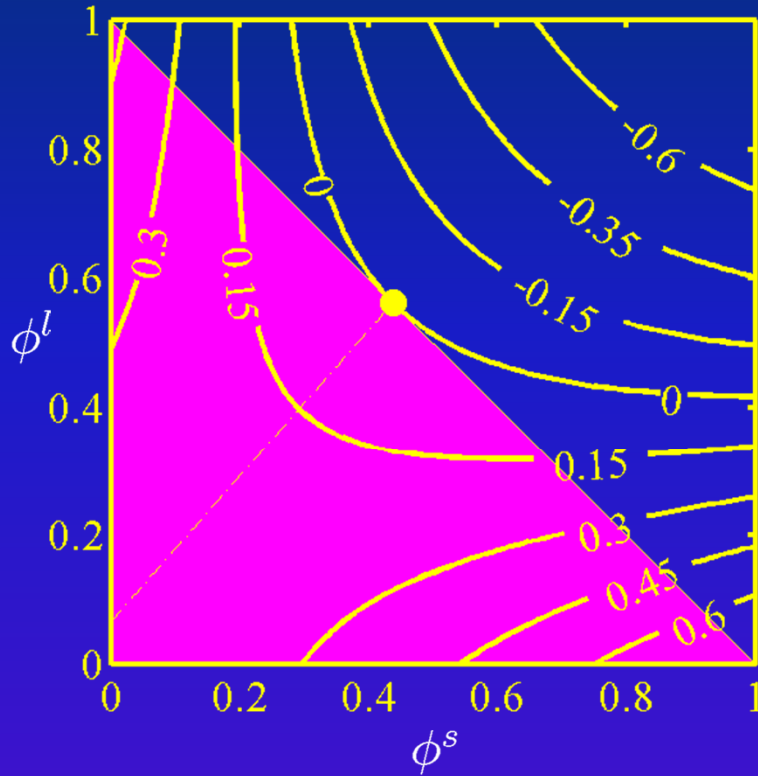- The characteristic wave speeds are given by

$$\det(A - \lambda I) = 0,$$

- The eigenvalue $\lambda = 0$ is easily spotted, which leaves
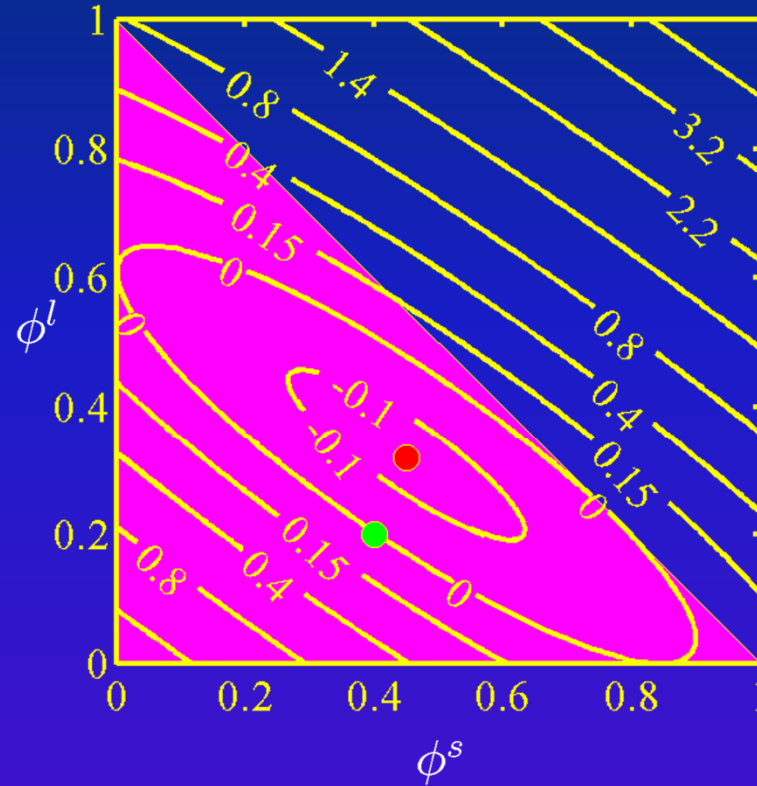
$$\lambda^2 + \gamma_1\lambda + \gamma_2 = 0,$$

- The two characteristic wave speeds are therefore

$$\lambda_{1,2} = (-\gamma_1 \pm \sqrt{\Delta})/2.$$

$S_{ls} = 1,\ S_{lm} = 0.3,\ S_{ms} = 0.1$
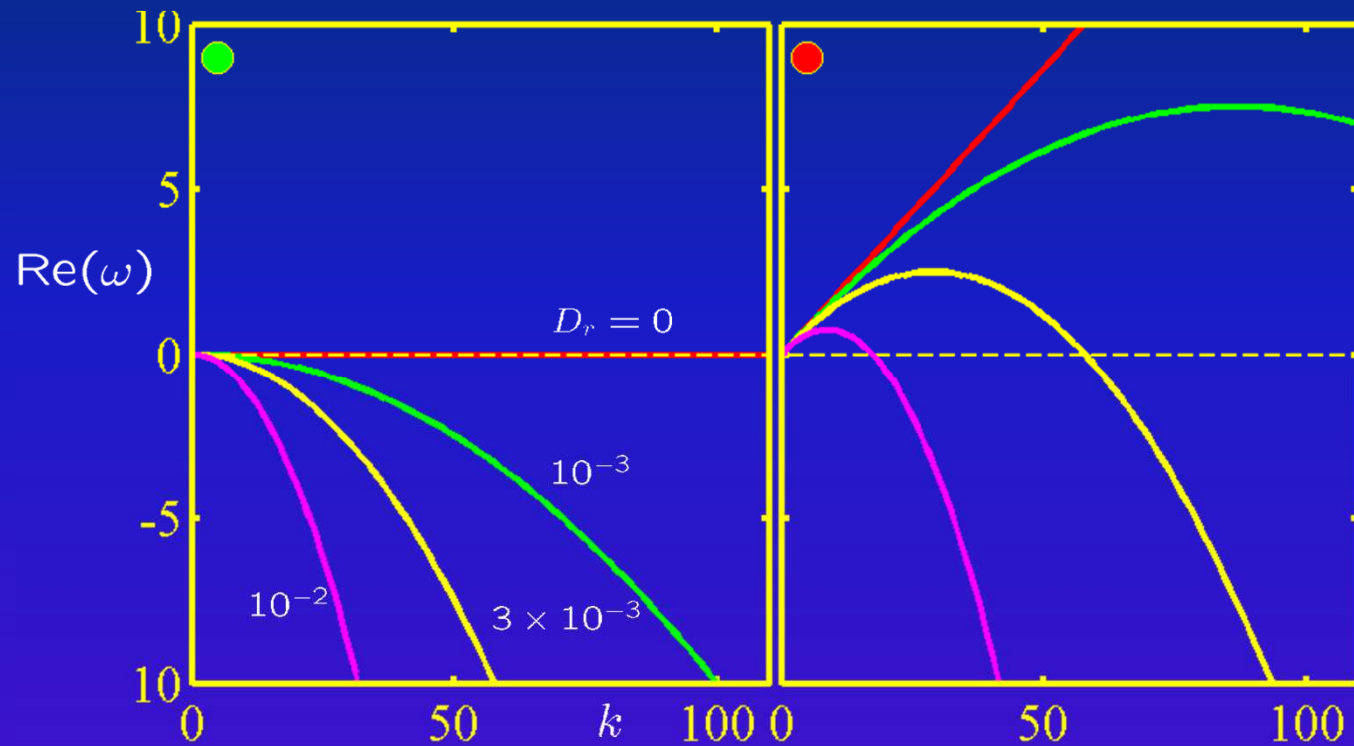
$S_{ls} = 1/8,\ S_{lm} = 1,\ S_{ms} = 3/8$

- The discriminant

$$\Delta \;=\; \big[(S_{ls} - 2S_{lm} - S_{ms})\phi^l + (S_{ls} - S_{lm} - 2S_{ms})\phi^s +$$
$$S_{lm} + S_{ms}\big]^2 - 4(S_{ls} - S_{ms})(S_{ls} - S_{lm})\phi^l \phi^s.$$

- When this is positive there are two real distinct roots and system is strictly hyperbolic

$\Omega_1:$	$S_{ms} \geq S_{ls} \geq S_{lm},$			hyperbolic
$\Omega_2:$	$S_{lm} \geq S_{ls} \geq S_{ms},$			hyperbolic
$\Omega_3:$	$S_{ls} > S_{lm}$	and	$S_{ls} > S_{ms},$	hyperbolic
$\Omega_4:$	$S_{ls} < S_{lm}$	and	$S_{ls} < S_{ms},$	looses hyperbolicity

20

Re(ω)

$D_r = 0$

10^{-3}

10^{-2} 3×10^{-3}

- Linear stability about a constant base state $\phi^\nu = \phi_0^\nu$

$$\phi^\nu = \phi_0^\nu + C_\nu \exp(ikz + \omega t)$$

- The largest real root

$$\mathrm{Re}(\omega) = \begin{cases} -D_r k^2, & \Delta_0 \geq 0, \\ -D_r k^2 + \dfrac{k}{2}\sqrt{-\Delta_0}, & \Delta_0 < 0. \end{cases}$$

- If $D_r = 0 \Rightarrow$ Hadamard unstable and ill-posed in Ω_4

- Homogeneous initial state with a small perturbation

$$
\begin{aligned}
\phi^l(z,0) &= \phi_0^l + 0.01\sin(2\pi nz), \\
\phi^s(z,0) &= \phi_0^s - 0.01\sin(2\pi nz),
\end{aligned}
$$

- Solve using a standard Galerkin finite element method

$$
\begin{aligned}
\frac{\partial \phi^l}{\partial t} + \frac{\partial}{\partial z}(S_{lm}\phi^l(1-\phi^l-\phi^s) + S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^l}{\partial z}\right) \\
\frac{\partial \phi^s}{\partial t} + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1-\phi^l-\phi^s)) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^s}{\partial z}\right)
\end{aligned}
$$
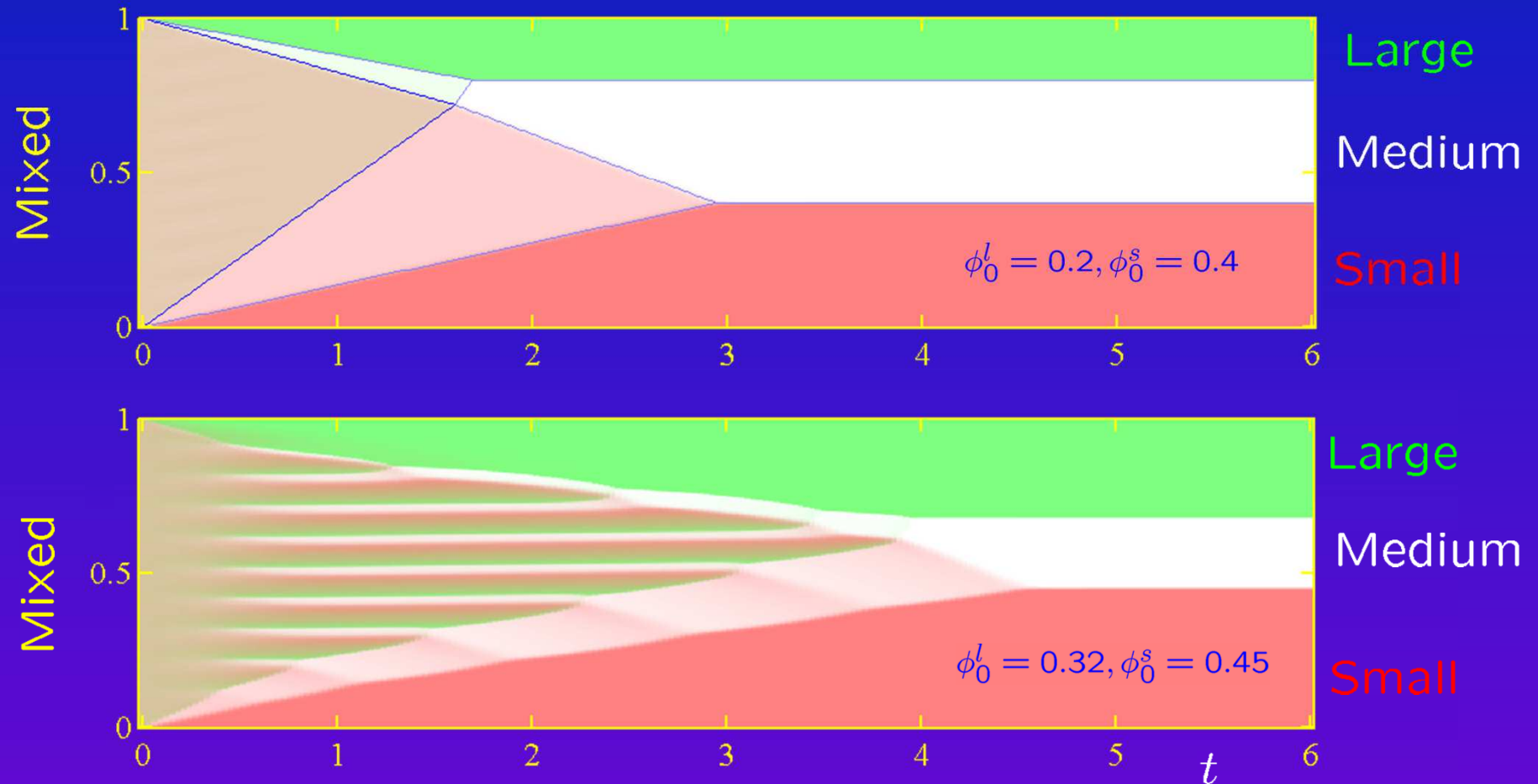
- subject to no-flux conditions

$$
\mathcal{F}^\nu = -\sum_{\forall \mu} S_{\nu\mu}\phi^\nu\phi^\mu + D_r\frac{\partial \phi^\nu}{\partial z} = 0, \quad \text{on} \quad z = 0, 1
$$

- using Matlab inbuilt function pdepe

Diffusive remixing regularizes the theory

Ω_4: $S_{ls} = 1/8$, $S_{lm} = 1$, $S_{ms} = 3/8$, $D_r = 10^{-3}$

Large

Medium

$\phi_0^l = 0.2, \phi_0^s = 0.4$

Small

Large

Medium

$\phi_0^l = 0.32, \phi_0^s = 0.45$

Small

t

- Instabilities still develop when $\Delta_0 < 0$
- BUT are annihilated after a finite distance

23

- Homogeneous inflow at $x = 0$

$$\phi^\nu(0, z) = \phi_0^\nu$$
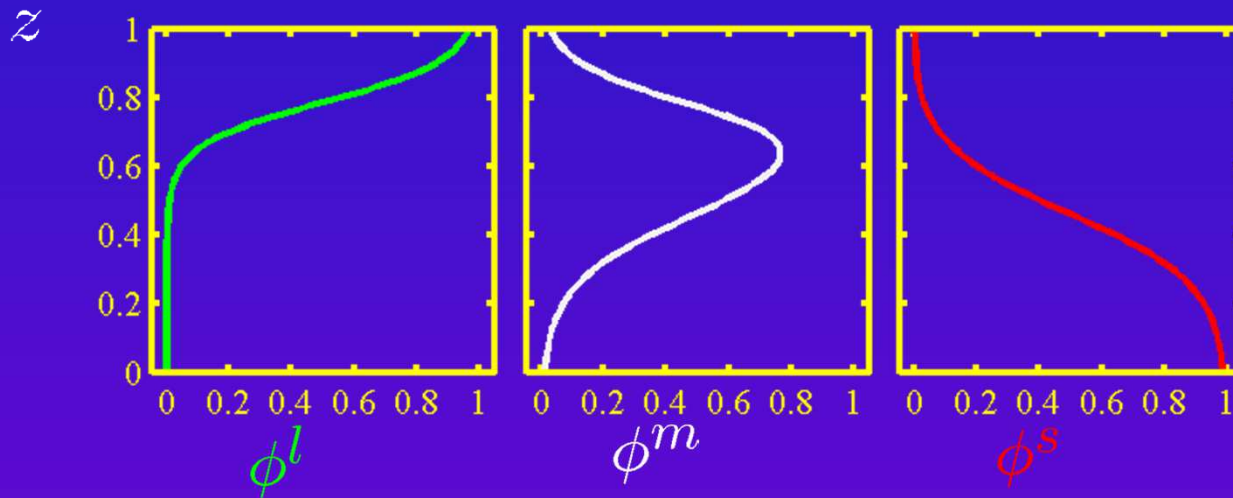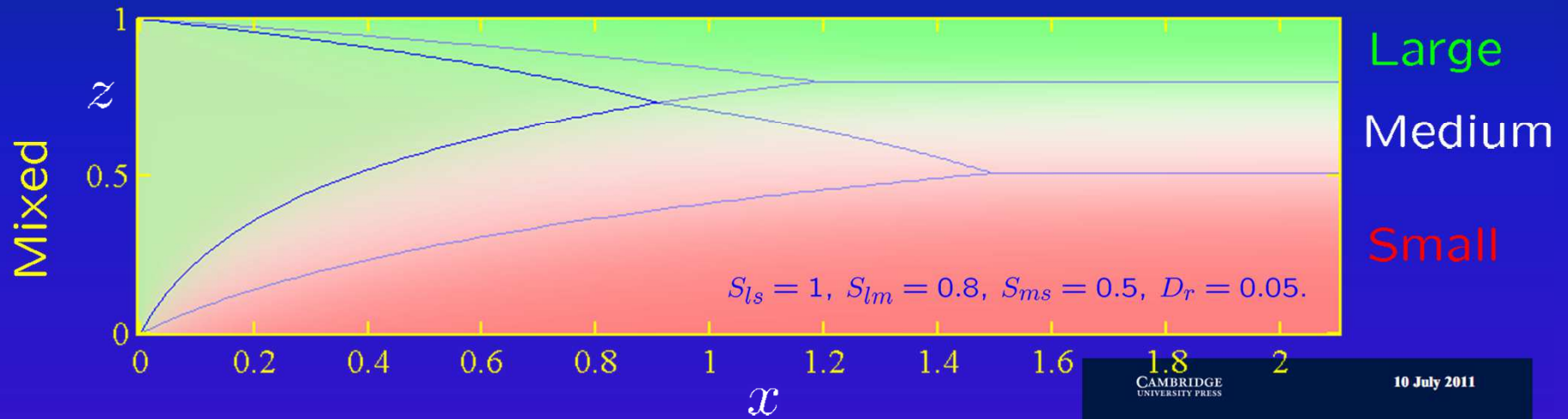
- with prescribed exponential downstream velocity field

$$u(z) = \frac{\beta \exp(\beta z)}{\exp(\beta) - 1}, \quad \beta > 0.$$

- Solve the two-dimensional steady problem

$$u\frac{\partial \phi^l}{\partial x} + \frac{\partial}{\partial z}(S_{lm}\phi^l(1 - \phi^l - \phi^s) + S_{ls}\phi^l\phi^s) = \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^l}{\partial z}\right)$$

$$u\frac{\partial \phi^s}{\partial x} + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1 - \phi^l - \phi^s)) = \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^s}{\partial z}\right)$$
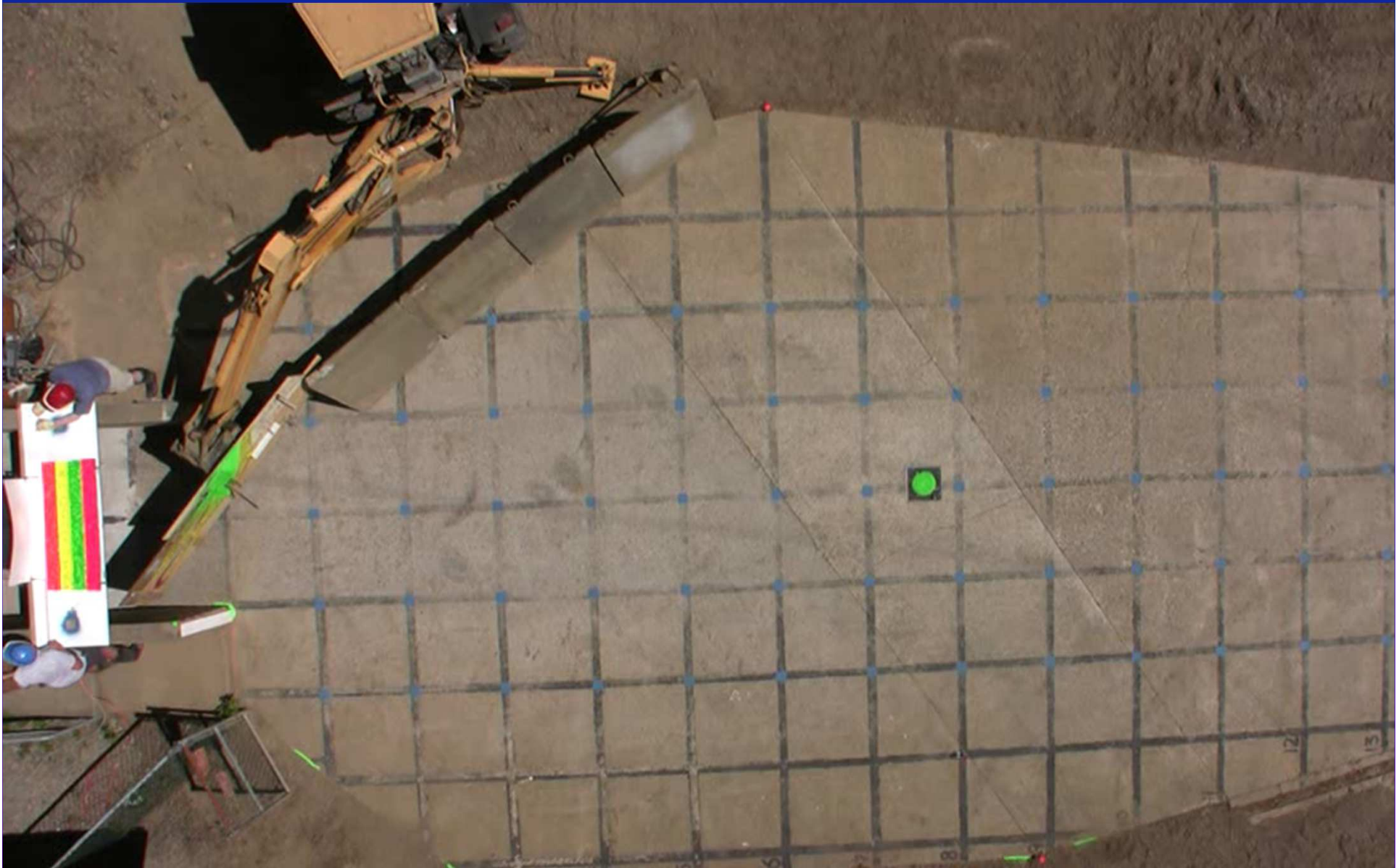
- subject to no-flux conditions at $z = 0, 1$
- using Matlab inbuilt function pdepe (Galerkin Method)

24

Reverse distribution grading

$$S_{ls} = 1, \ S_{lm} = 0.8, \ S_{ms} = 0.5, \ D_r = 0.05.$$

Large

Medium

Small

Mixed

z

x

z

ϕ^l ϕ^m ϕ^s

CAMBRIDGE
UNIVERSITY PRESS   10 July 2011

Journal of
Fluid Mechanics
VOLUME 678

- Non-distructive experiments are hard.
- Can one use X-Ray tomography to reconstruct evolution?

25

Transport and accumulation of large particles
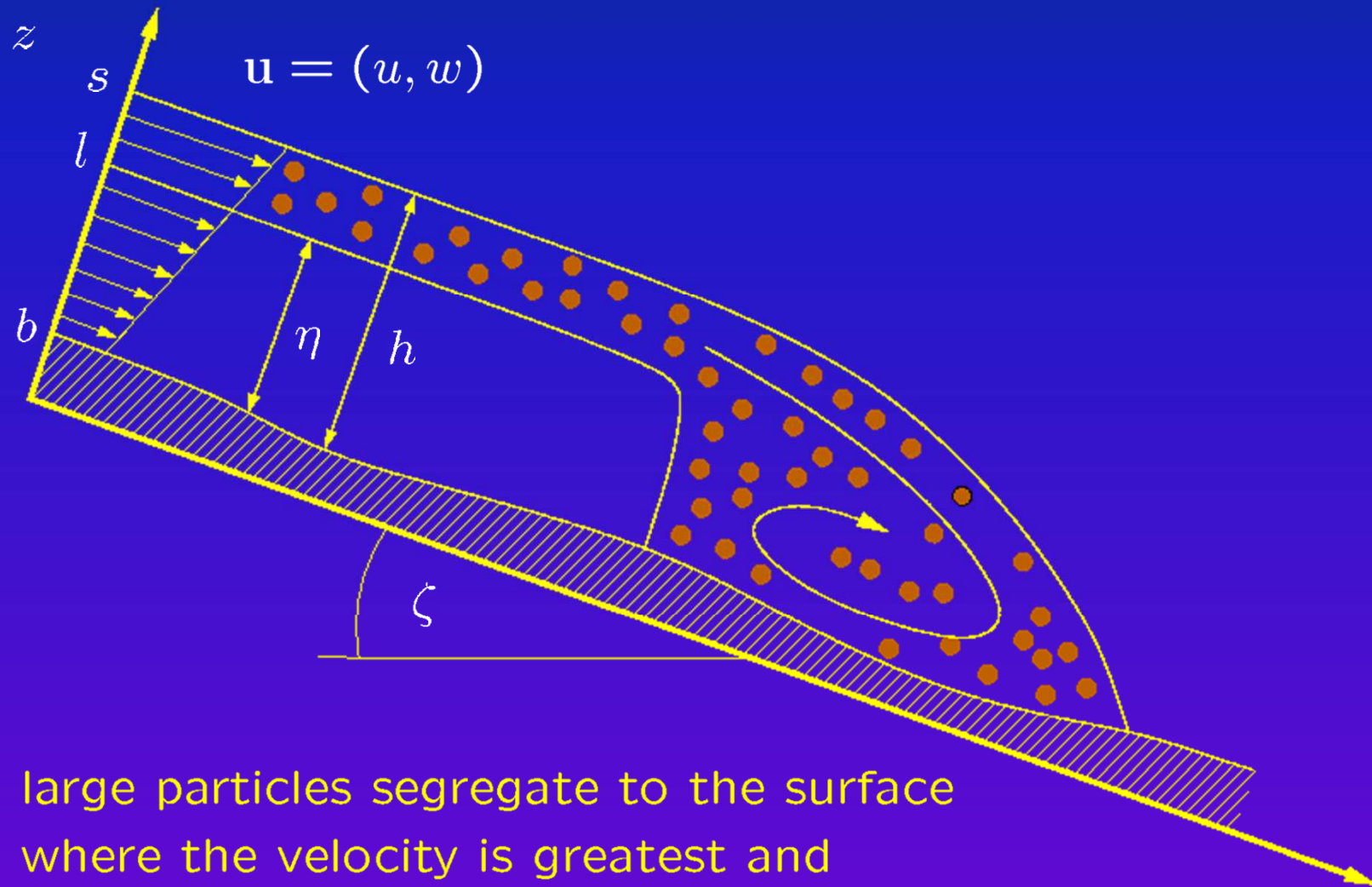
- large particles segregate to the surface
- where the velocity is greatest and
- are transported to the flow front where they are
- over run and recirculated by particle size segregation

A depth averaged theory for particle size segregation

- Integrating the segregation-remixing equation w.r.t z
- subject to the no flux and kinematic boundary conditions gives

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\overline{\phi u}) = 0.$$

where the depth-averaged concentration and flux are

$$\bar{\phi} = \frac{1}{h}\int_b^s \phi\,dz \quad \text{and} \quad \overline{\phi u} = \frac{1}{h}\int_b^s \phi u\,dz,$$

- Stratification experiments suggest

    - avalanches segregate rapidly by size

    - the segregation can be very sharp

    - that there is velocity shear through the avalanche depth

30

- Assuming the velocity profile is linear with depth and
- the particle size distribution is always sharply inversely graded

$$u = \alpha \bar{u} + 2(1 - \alpha)\bar{u}\left(\frac{z - b}{h}\right), \quad \phi = \left\{ \begin{array}{ll} 0, & l < z \leq s, \\ 1, & b \leq z < l. \end{array} \right.$$

- implies

$$h\bar{\phi} = \eta, \qquad h\overline{\phi u} = \eta\bar{u} - (1 - \alpha)\bar{u}\eta\left(1 - \frac{\eta}{h}\right)$$

- substituting into the depth-averaged segregation equation
- yields the large particle transport equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta\bar{u}) - \frac{\partial}{\partial x}\left((1 - \alpha)\bar{u}\eta\left(1 - \frac{\eta}{h}\right)\right) = 0.$$

- for the evolution of the inversely graded shock interface η.

31

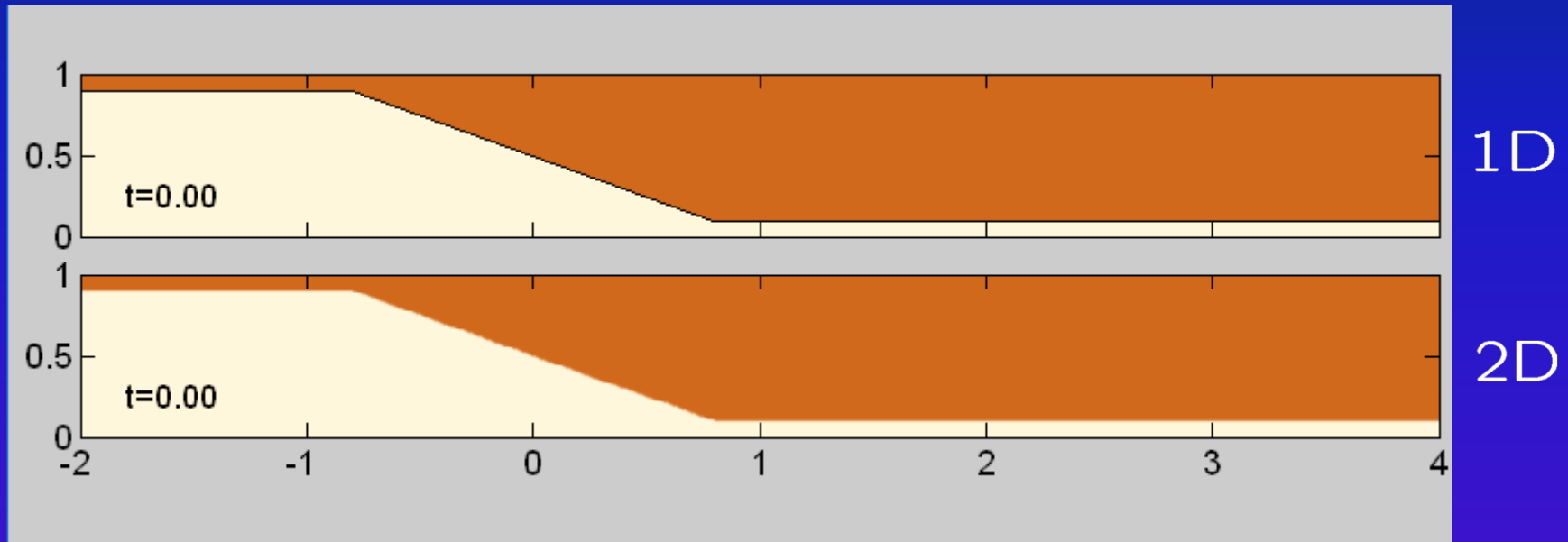- Using $\eta = h\bar{\phi}$ this can also be rewritten as

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\bar{\phi}\bar{u}) - \frac{\partial}{\partial x}\left((1-\alpha)h\bar{u}\bar{\phi}\left(1-\bar{\phi}\right)\right) = 0.$$

- Remarkably similar to the segregation equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial z}(\phi w) - S_{ls}\frac{\partial}{\partial z}\left(\phi(1-\phi)\right) = \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi}{\partial z}\right)$$

- from which it is derived. We therefore know

    - large grains transported forwards $\Rightarrow$ bouldery flow front

    - small grains transported backwards $\Rightarrow$ fines rich tail

Formation of a breaking size segregation wave

1D

2D

- once the grains have segregated into inversely graded layers
- solution is the same as the full theory

  Gray, Shearer & Thornton (2006) *Proc. Roy. Soc.* **462**, 947-972
  Shearer *et al.* (2008) *European J. Applied Math.* **19**, 61-86.

- except when the interface breaks and
- a jump forms instead of a breaking size segregation wave
- Net large/small particle transport is exactly the same.

  Thornton & Gray (2008) *J. Fluid Mech.* **596**, 261-284

Large particle accumulation at an avalanche flow front
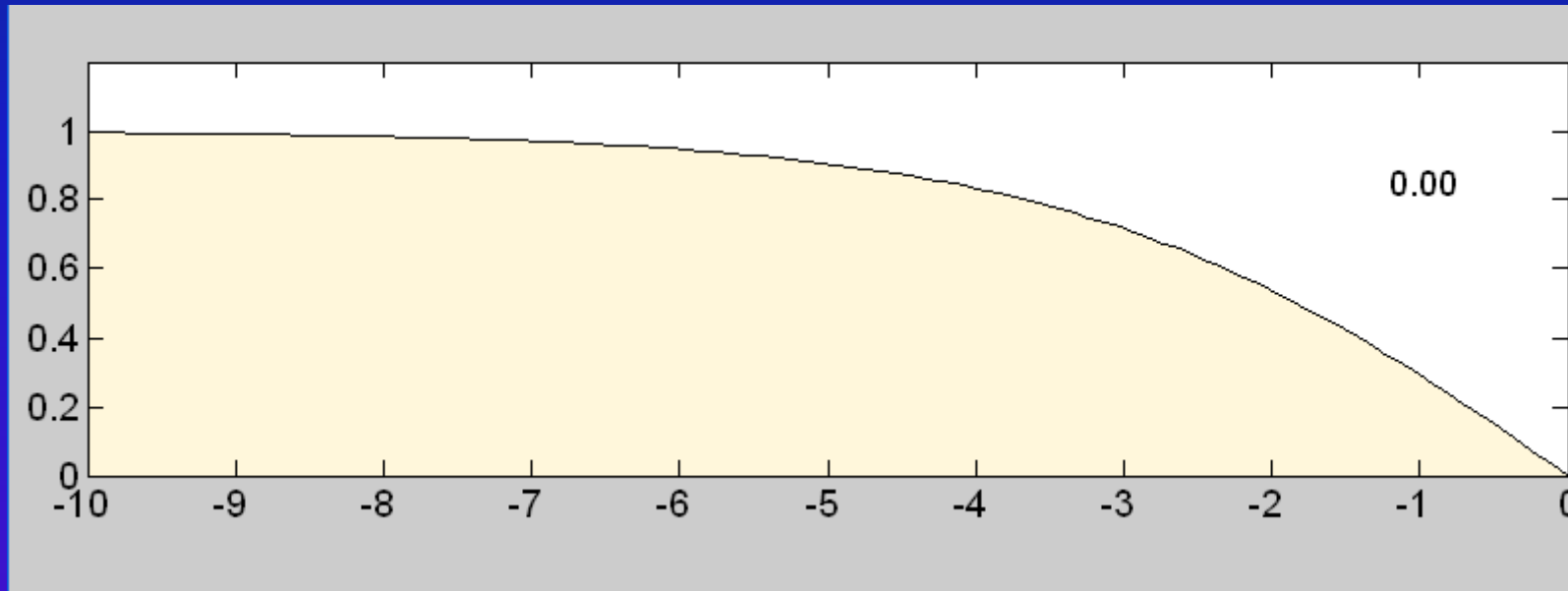
$$\xi = x - u_F t$$

- Initially the avalanche front is composed of fines
- large particle are transported to the flow front
- where they are over run and are recirculated

- inversely graded avalanche connected to large rich front
- by shock whose position is found by a shock fitting procedure

Can depth-averaged models capture finger formation?

- For avalanche thickness h, small particle thickness η and depth-averaged velocity $\overline{u}$ the 2D coupled model is

$$\frac{\partial h}{\partial t} + \text{div}(h\overline{u}) = 0,$$

$$\frac{\partial \eta}{\partial t} + \text{div}\left(\eta\overline{u} - (1-\alpha)\eta\left(1 - \frac{\eta}{h}\right)\overline{u}\right) = 0,$$

$$\frac{\partial}{\partial t}(h\overline{u}) + \text{div}(h\overline{u} \otimes \overline{u}) + \text{grad}\left(\frac{1}{2}h^2\cos\zeta\right) = h\boldsymbol{S},$$
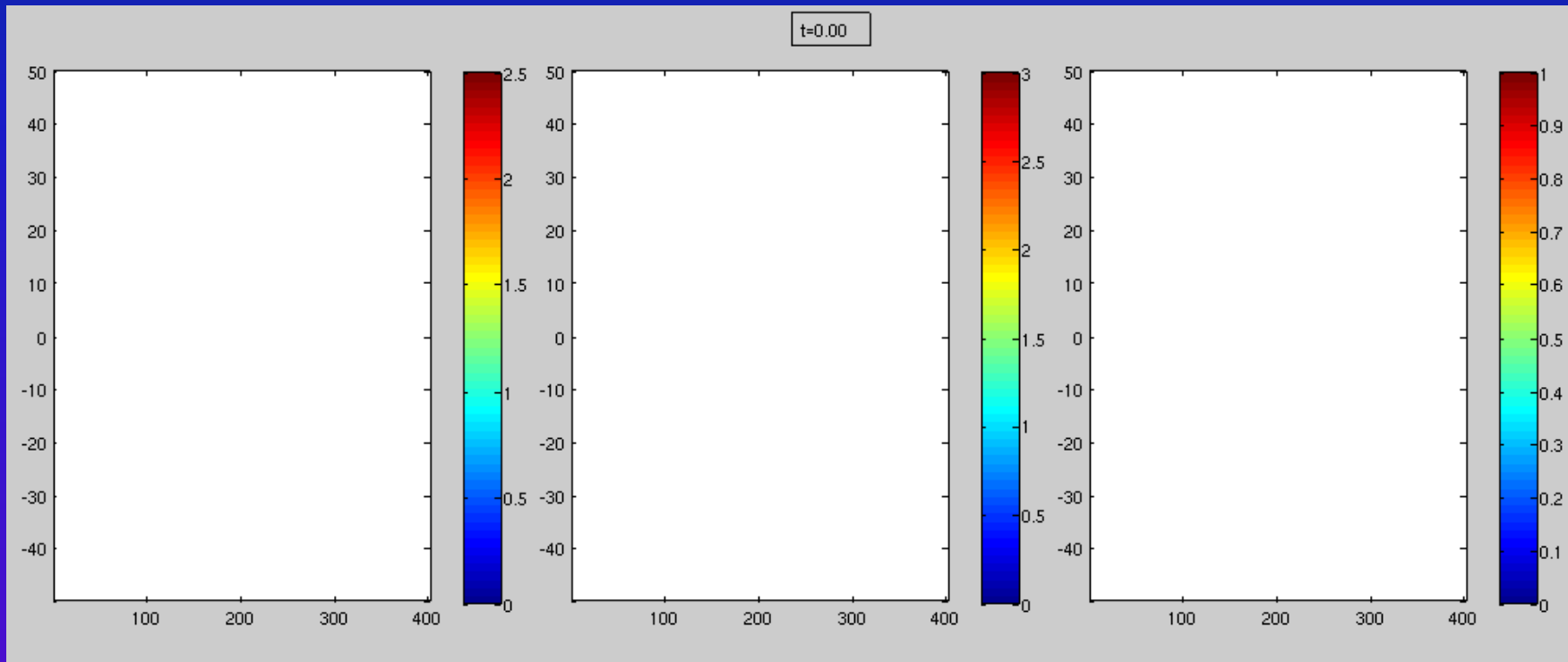
- source terms composed of gravity and basal friction

$$\boldsymbol{S} = \begin{pmatrix} \sin\zeta - \mu(\overline{u}/|\overline{u}|)\cos\zeta, \\ -\mu(\overline{v}/|\overline{u}|)\cos\zeta, \end{pmatrix}$$

- coupling through $\overline{\phi} = \eta/h$ dependent friction coefficient

$$\mu = \left(1 - \overline{\phi}\right)\mu^L + \overline{\phi}\mu^S$$

36

• Depth averaged coupled simulations ...

Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.

h $\qquad$ $|\overline{u}|$ $\qquad$ $\overline{\phi}$

- captures the instability mechanism, BUT model is too simple
- ... as it is ill-posed when $\mathsf{Fr} = (1-\alpha)^{-1}(2\phi-1)^{-1}$
- dissipation is sufficient to make it well-posed!
- BUT what form should rheology take?

Segregation-mobility feedbacks lead the formation of bouldery flow fronts, coarse grained levees and longer run-outs ...

Mt St Helens July 22, 1980

Mt St Helens July 22, 1980