From particles to continuum theory
Jamming, relaxation, anisotropy, and TIME!

Stefan Luding, MSM, CTW, UTwente, NL

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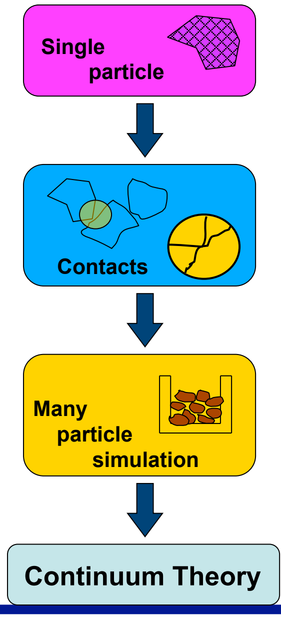
Overview

- Introduction
- Contact models
- Many particle simulation
- Local coarse graining
- Continuum Theory
- … Anisotropy
- … Time-scales



Continuum theory

mass conservation:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

momentum conservation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) = -\frac{\partial}{\partial x_i}P + \frac{\partial}{\partial x_j}\sigma_{ij}^{\mathrm{dev}} + \rho g_i$$

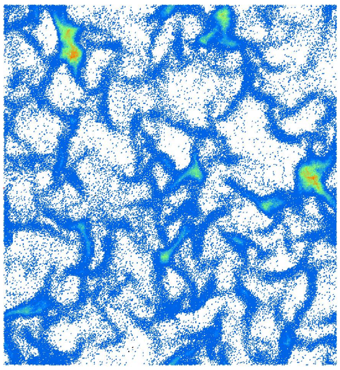
energy balance:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho v^2\right) = -\frac{\partial}{\partial x_k}\left[\rho u_k\left(\frac{P}{\rho} + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) - u_i\sigma_{ik}^{\mathrm{dev}} - K\frac{\partial}{\partial x_k}\left(\tfrac{1}{2}\rho v^2\right)\right] + \rho u_i g_i - I$$

- **Pressure *P***
- **Shear Stress** $\sigma_{ij}^{\mathrm{dev}}$
- **Energy Dissipation Rate *I***

How to understand clustering ?

Goldhirsch, Zanetti 1993, …



- Higher density
- More dissipation
- Lower Pressure
- etc.

… why ?

dissipation = energy loss (irreversible)

Freely cooling system

homogeneous steady state: $\frac{\partial}{\partial x_i} = 0 \qquad g_i = u_i = 0$

mass & momentum conservation – OK

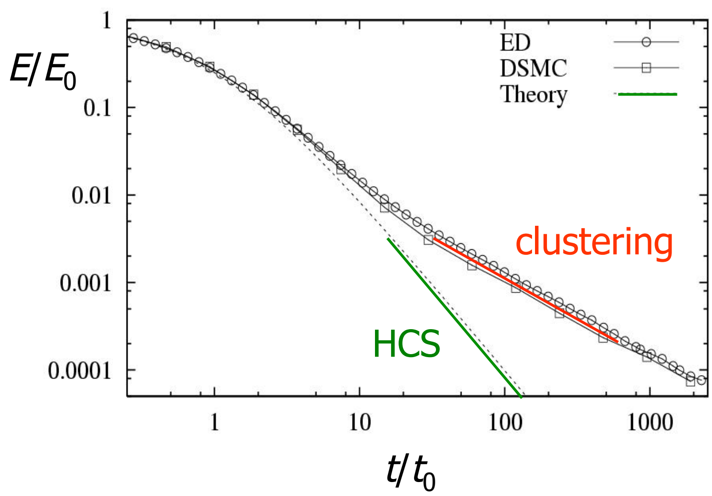
energy balance: $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -I \qquad I \propto \rho\left(1-r^2\right)v^3$

mean field (MF) solution:

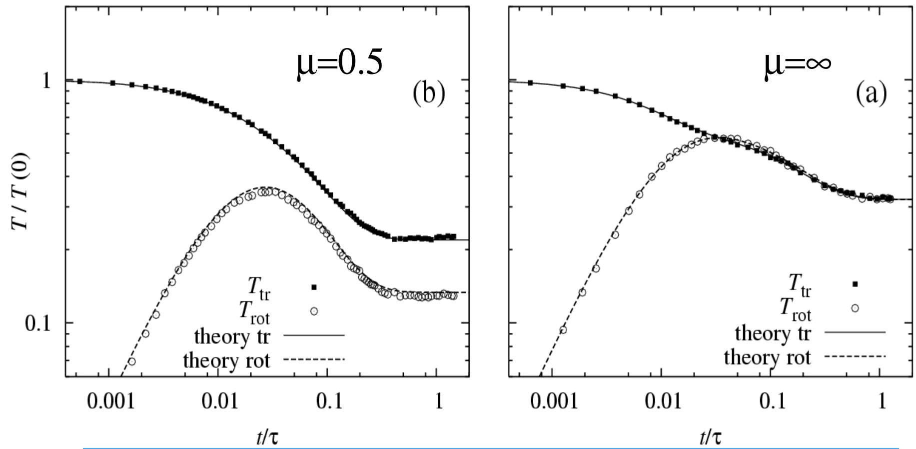
$$\frac{v}{v_0} = \frac{1}{1+\alpha\left(1-r^2\right)v_0 t}$$

$$\frac{E}{E_0} = \frac{1}{\left(1+\alpha\left(1-r^2\right)v_0 t\right)^2}$$

Freely cooling system (HCS)



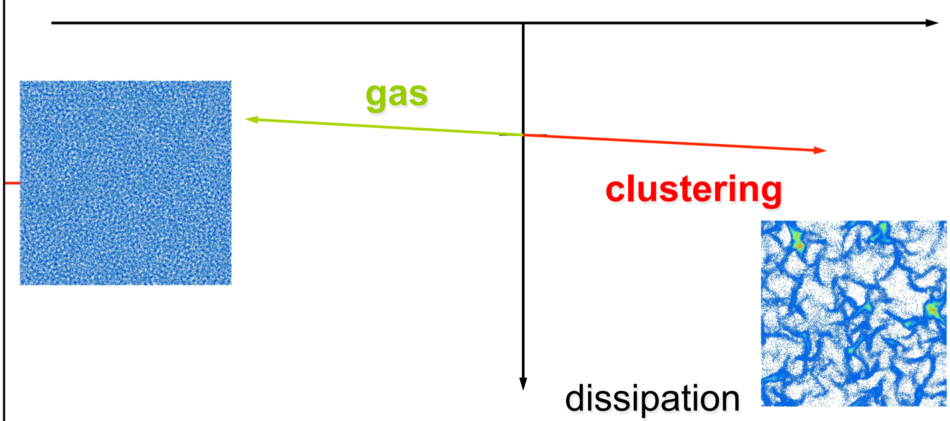
Kinetic theory with Coulomb friction



… possible, but serious hard work …

NO shortcut

Clustering/Agglomeration



Elastic hard spheres

elastic steady state: $\frac{\partial}{\partial t} = 0$ $u_i = I = 0$

mass & energy conservation – OK

momentum balance: $0 = -\frac{\partial}{\partial x_i} P$ $g_i = 0$

- **Pressure *P***
- **Shear Stress** $\sigma_{ij}^{\text{dev}} = 0$
- **Energy Dissipation Rate *I=0***

First example ... pressure

$$P = \frac{E}{V}\left(1 + (1+r)\nu g_{2a}(\nu)\right)$$

$$g_{2a}(\nu) = ?$$

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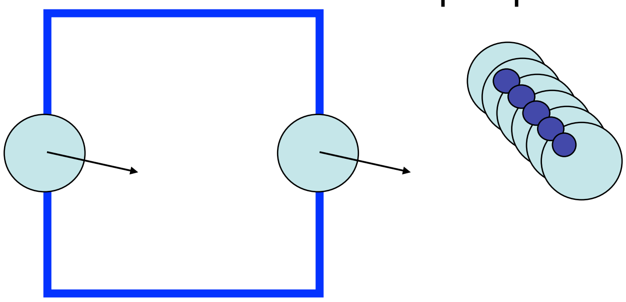
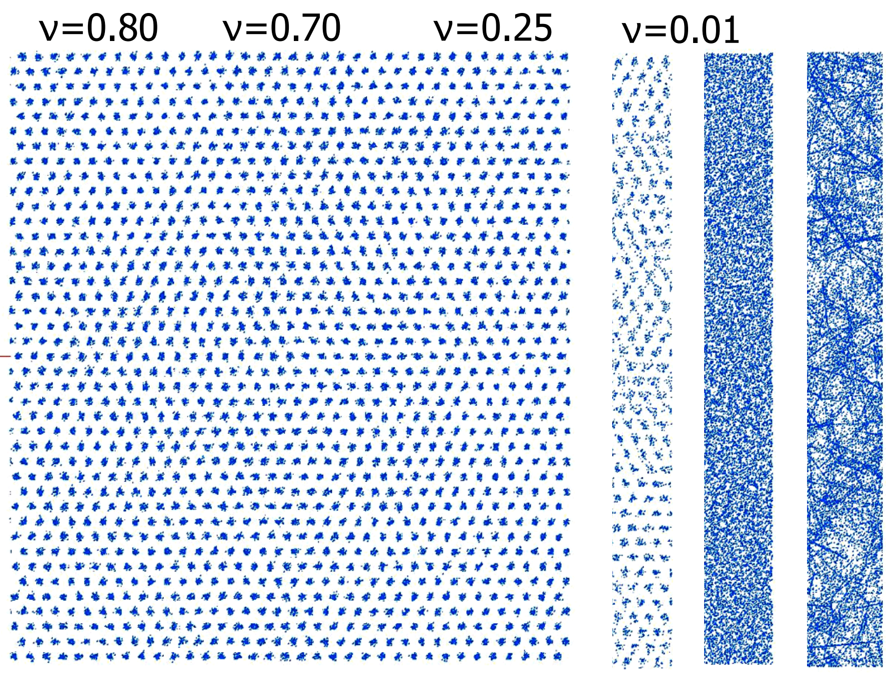
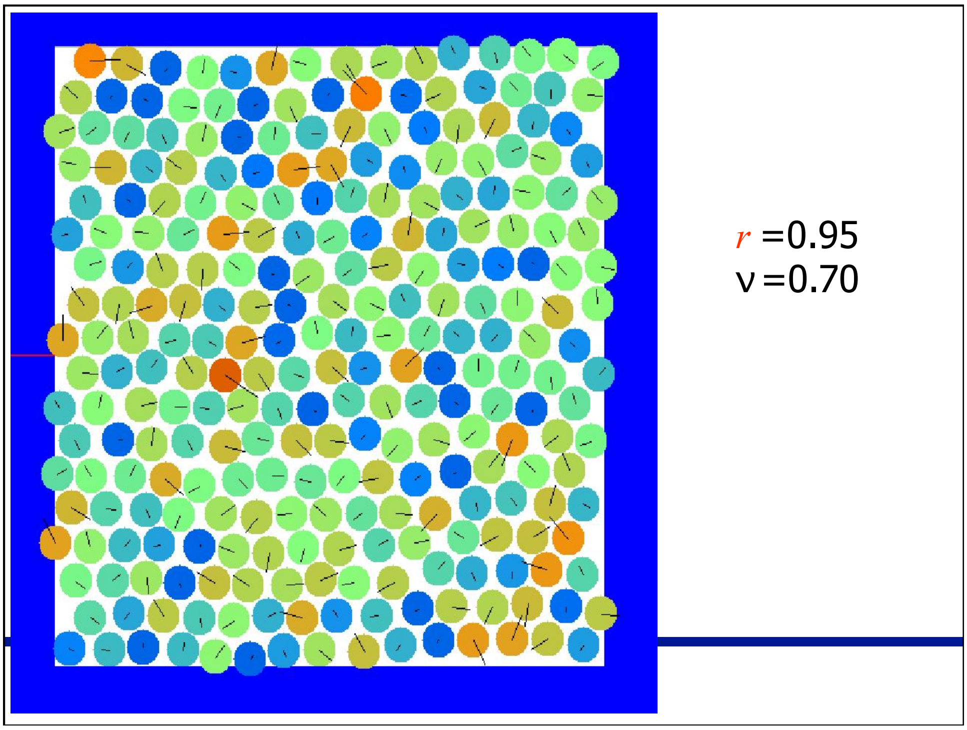
Elastic Hard Sphere Model

simulate N particles
in a peridoc box

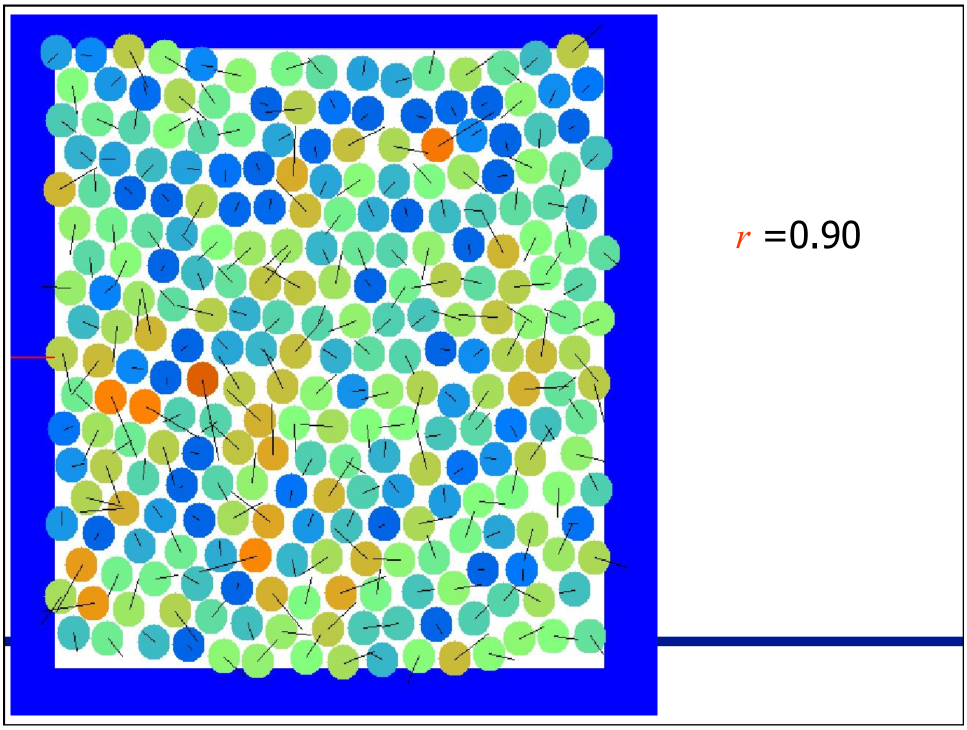
… plot path-lines



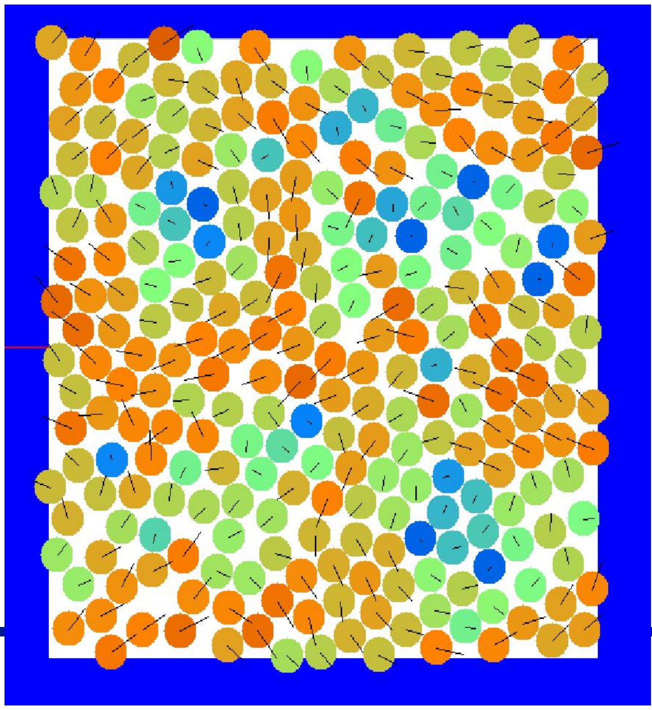




r =0.95
ν =0.70

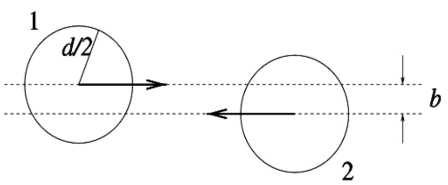


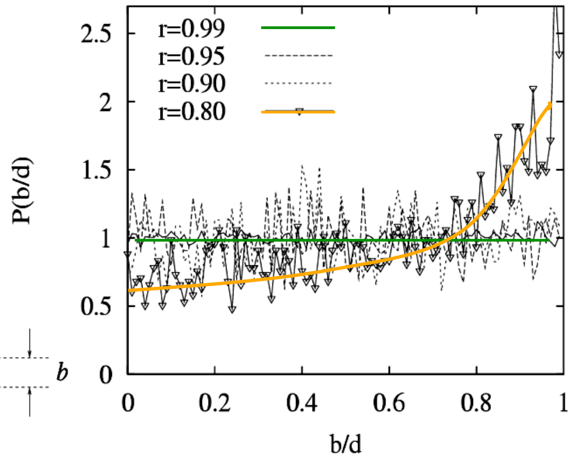
r =0.90



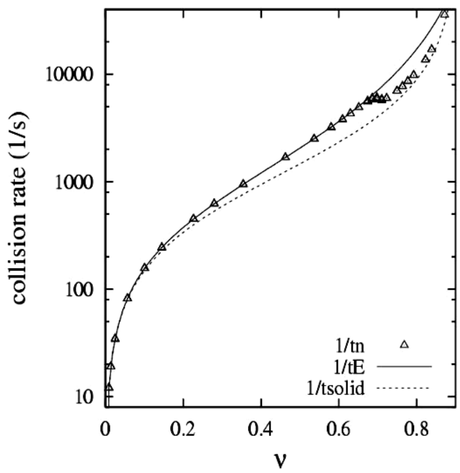
r =0.80

Collision parameter

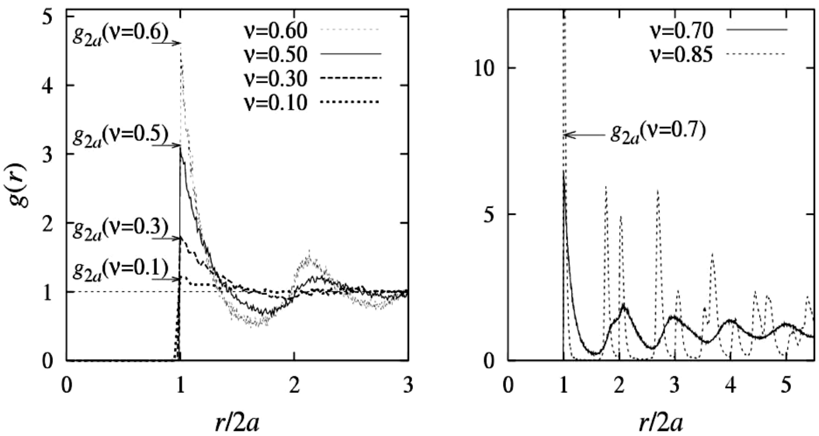




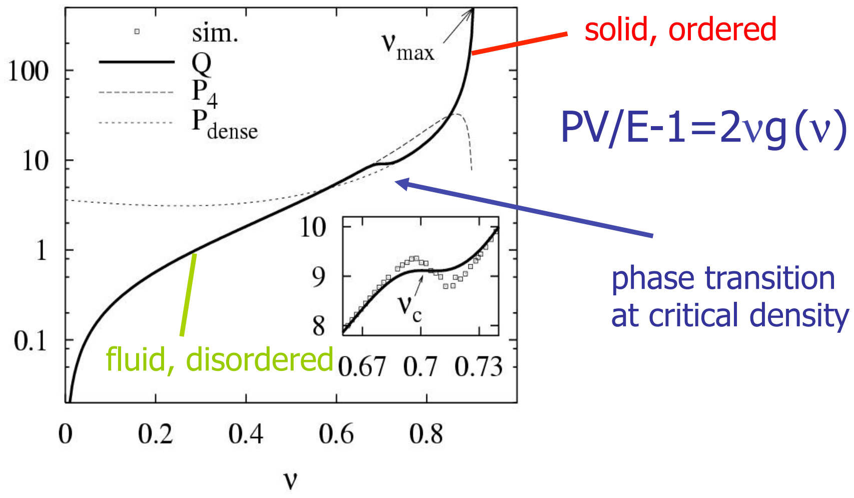
Collision rate – time scale



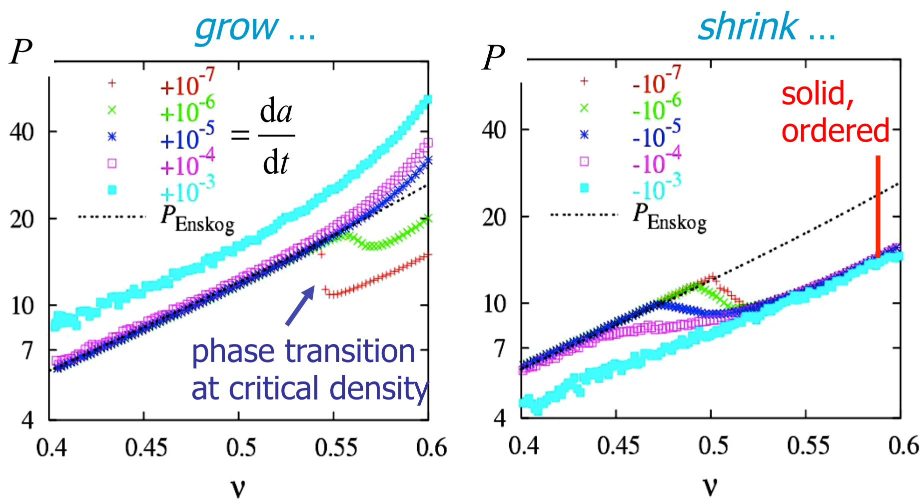
Contact probability – correlation function



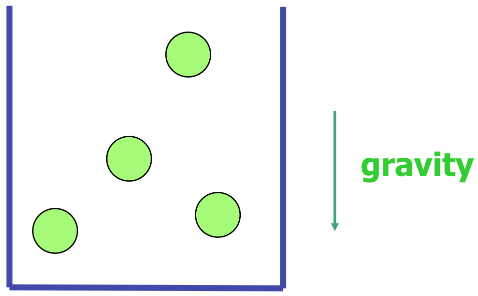
Pressure (Equation of State – 2D)



Pressure (Equation of State – 3D)

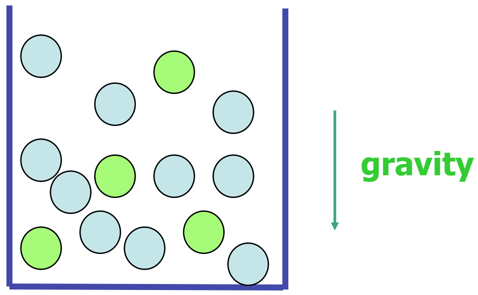


Elastic hard spheres in gravity



- N particles
- Kinetic Energy
- What is the ***density profile*** ?

Elastic hard spheres in gravity



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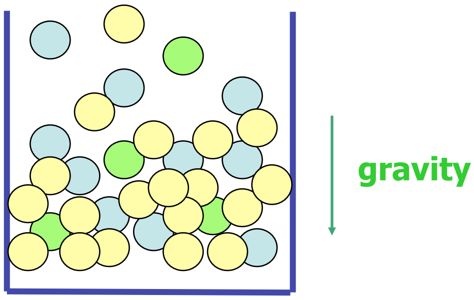
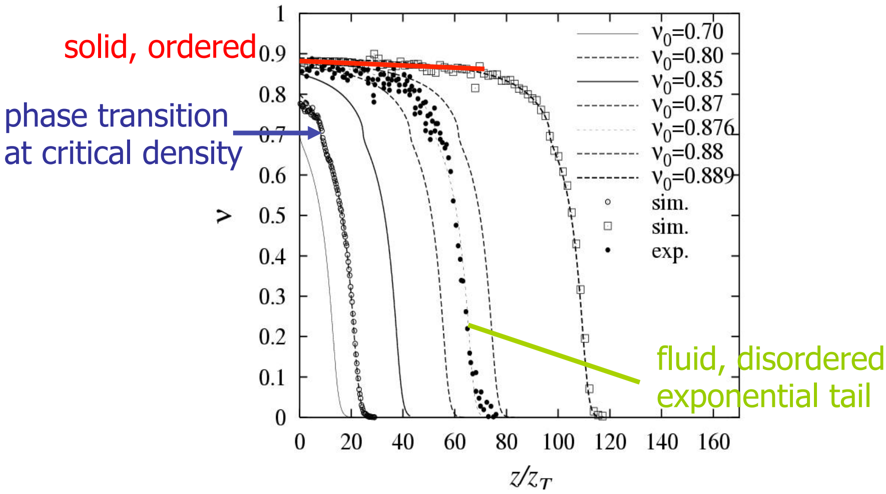
Elastic hard spheres in gravity



- **N particles**
- **Kinetic Energy**
- **What is the *density profile* ?**

Hard sphere gas in gravity



Continuum theory

steady state …

momentum conservation: $0 = -\frac{\partial}{\partial x_i} P + \rho g_i$

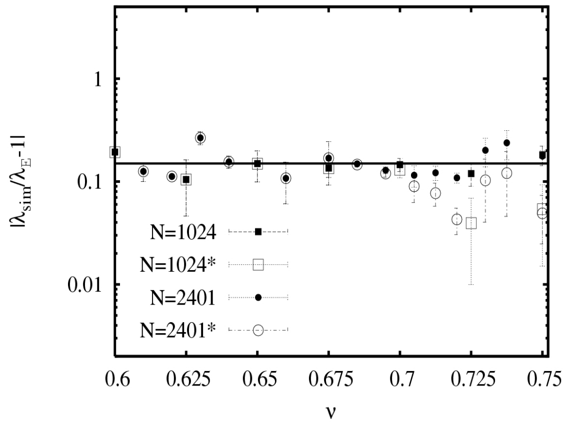
energy balance: $0 = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = -\frac{\partial}{\partial x_k}\left[-K\frac{\partial}{\partial x_k}\left(\tfrac{1}{2}\rho v^2\right)\right] - I$

- **Pressure *P***
- **Shear Stress** $\sigma_{ij}^{\mathrm{dev}} = 0$
- **Energy Dissipation Rate *I***

… heat-conductivity

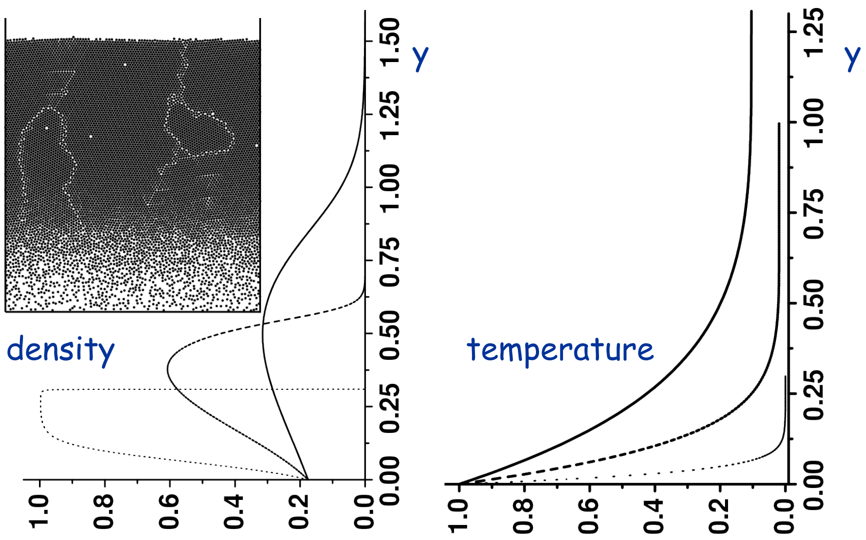
$$K = K\left(g_{2a}(\nu)\right)$$

Heat conductivity



Global equation of state for phase-transition

Density inversion: Results from T. Pöschel and B. Meerson



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… shear-viscosity

$$\eta = \eta\left(g_{2a}\left(\nu\right)\right)?$$

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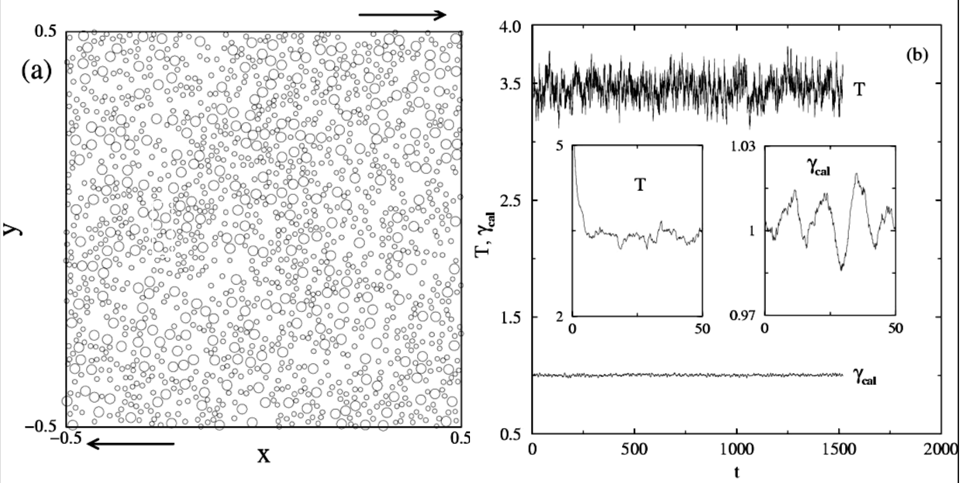
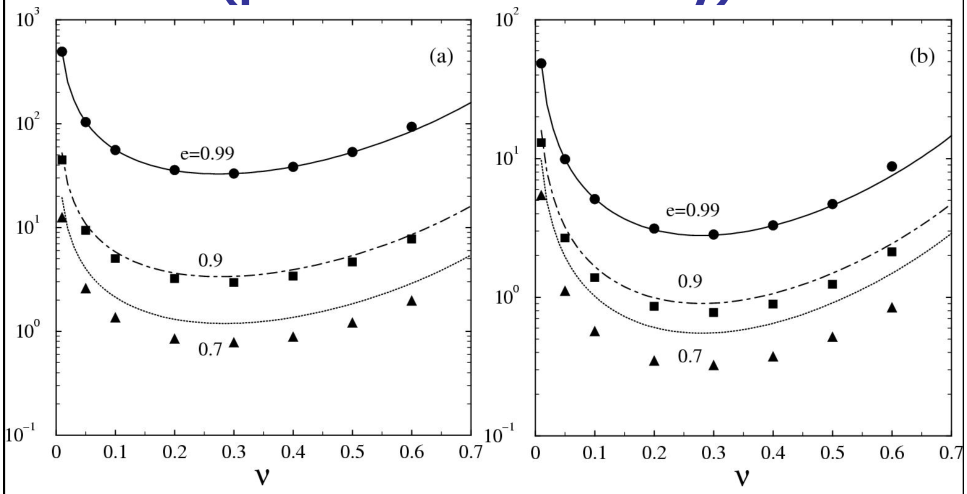
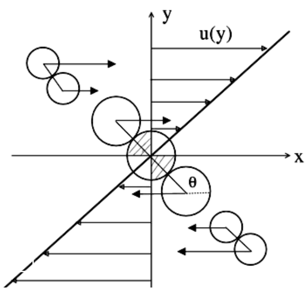
Shear (energy and rate)

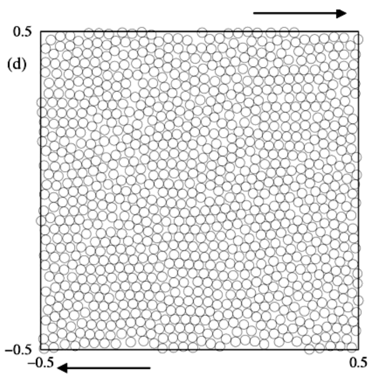


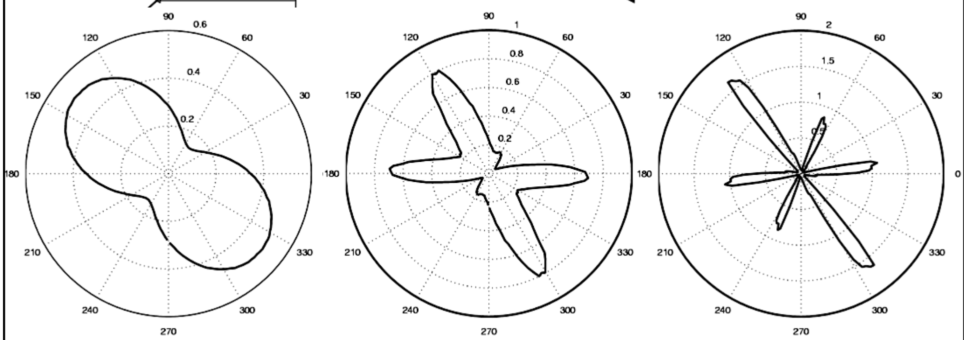
Shear (pressure and viscosity)



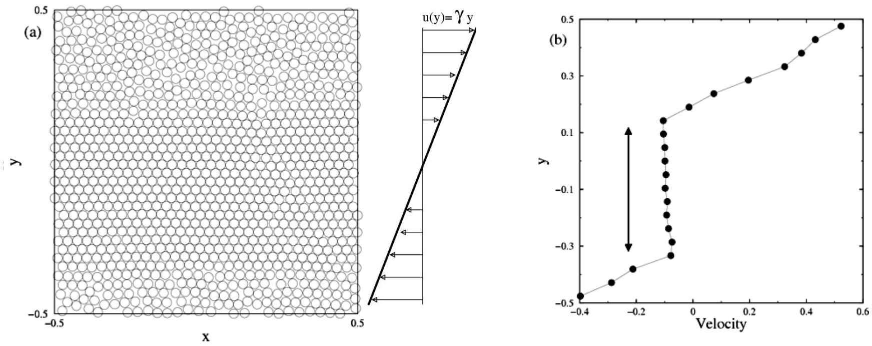
Structure formation







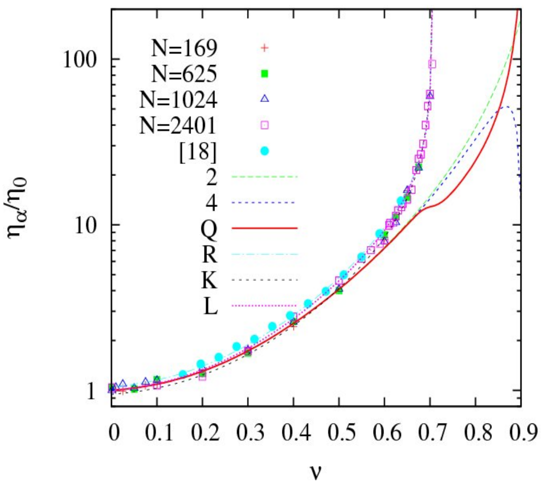
Structure formation



Low density -> linear velocity profile

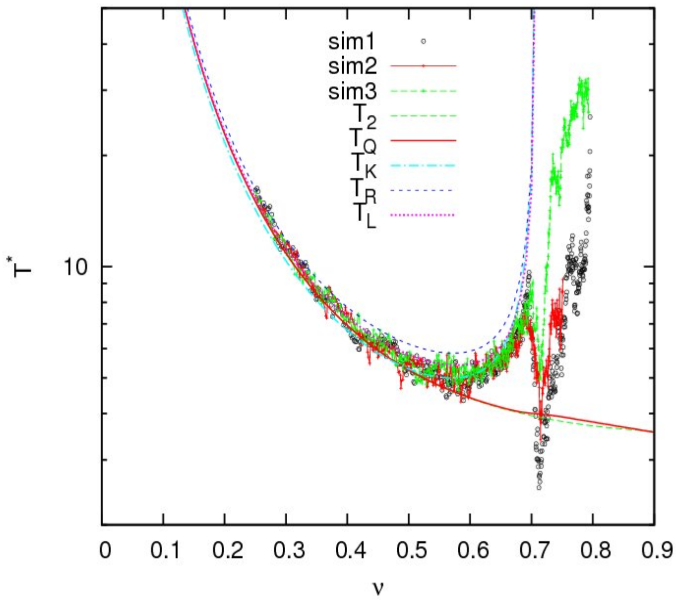
High density -> shear localization

shear “viscosity” (2D)

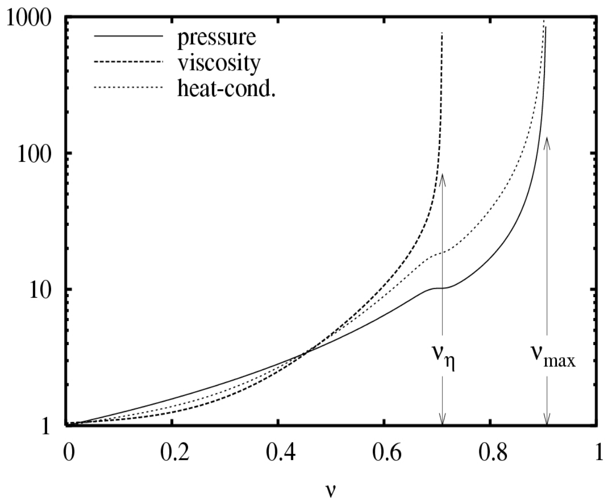


S. Luding, *Nonlinearity*, Dec. 2009

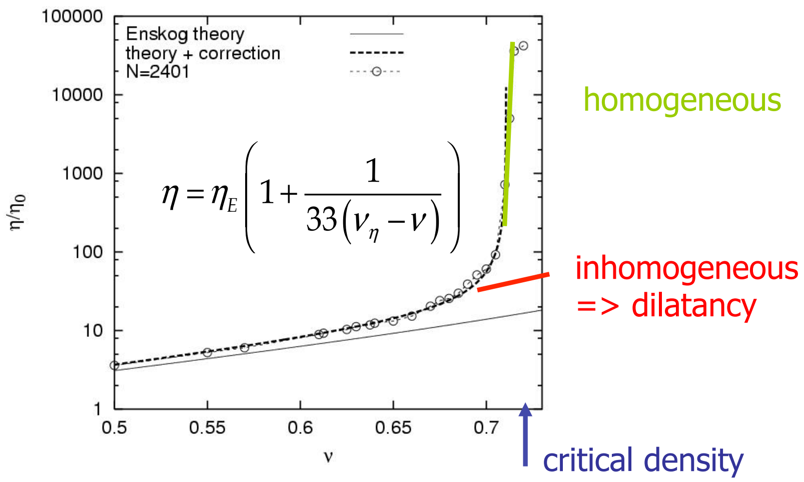
Sheared systems r=0.995



Global equations of state (2D)



Shear (viscosity at high density)



R. Garcia-Rojo, S. Luding, J. J. Brey, PRE 2006

Summary

- Pressure vs. density
  - Global equation of state (crystallization)
- Shear stress (viscosity) divergence -> J
  - Homogeneous and sheared …

Summary

- Pressure vs. density
  - Global equation of state (crystallization)
- Shear stress (viscosity) divergence -> J
  - Homogeneous and sheared
    - **But: which power law is it?**

Approach to jamming

Which power law is it? … really -1?

Approach to jamming

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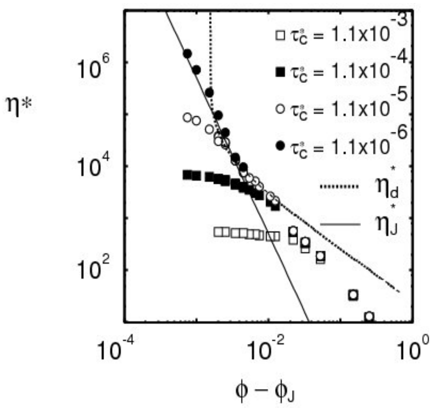
Otsuki, Hayakawa -> -3 !!!
Pouliquen -> -2 !!!

Approach to jamming

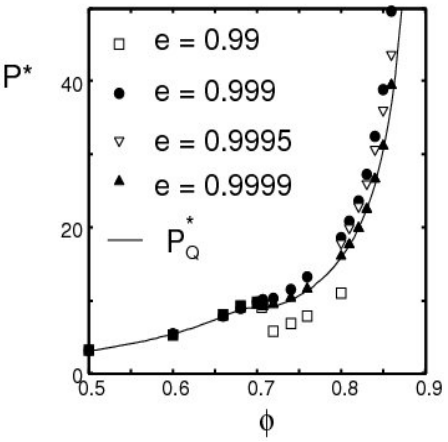
Which power law is it? … really -1?

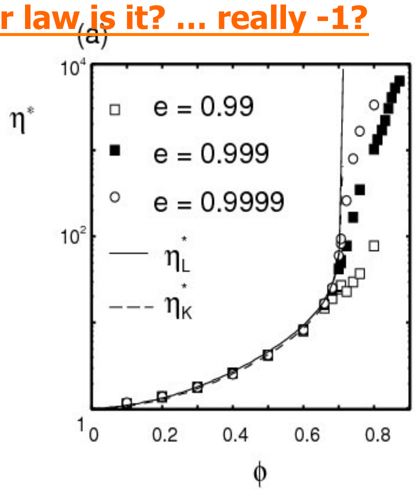


Otsuki, Hayakawa -> -3 !!!

Approach to jamming

- Which power law is it? … really -1?

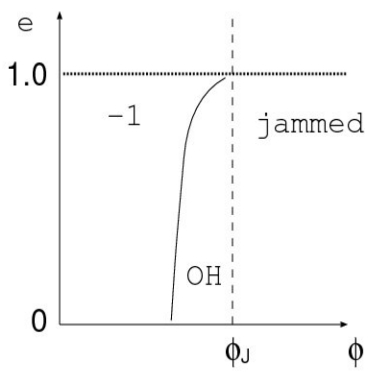


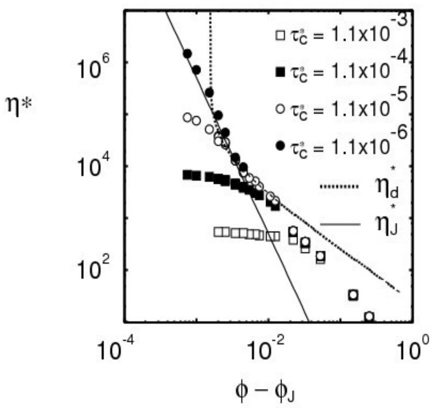


- control parameter -> dim.less. dissip.rate

Approach to jamming

- Which power law is it? … really -1?





M. Otsuki, H. Hayakawa, S. Luding, JTP, 2010

Summary (in between)

- Pressure vs. density
  - Global equation of state (crystallization)
- Shear stress (viscosity) divergence -> J
  - Homogeneous and sheared
    - **Which power law is it? Hard vs. Soft**
- Hard/soft jamming
  - Almost elastic vs. dissipative
  - Hard/rigid vs. soft
  - Kinetic theory vs. multi-particle contacts

Time-scales

- Contact duration tc
- Inverse shear rate
- Time between collisions
- Inverse dissipation rate
- (gravity)
- (pressure)

Continuum theory

mass conservation:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

momentum conservation:

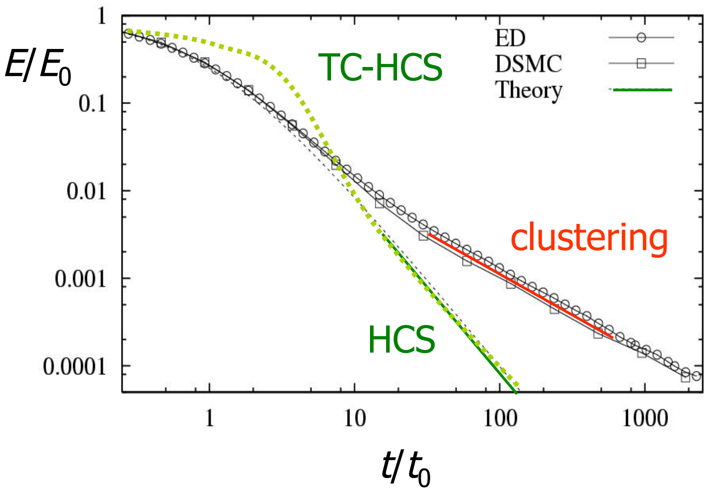
$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) = -\frac{\partial}{\partial x_i}P + \frac{\partial}{\partial x_j}\sigma_{ij}^{\text{dev}} + \rho g_i$$

energy balance:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho v^2\right) = -\frac{\partial}{\partial x_k}\left[\rho u_k\left(\frac{P}{\rho} + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) - u_i\sigma_{ik}^{\text{dev}} - K\frac{\partial}{\partial x_k}\left(\tfrac{1}{2}\rho v^2\right)\right] + \rho u_i g_i - I$$

- **Pressure** P
- **Shear Stress** σ_{ij}^{dev}
- **Energy Dissipation Rate** I

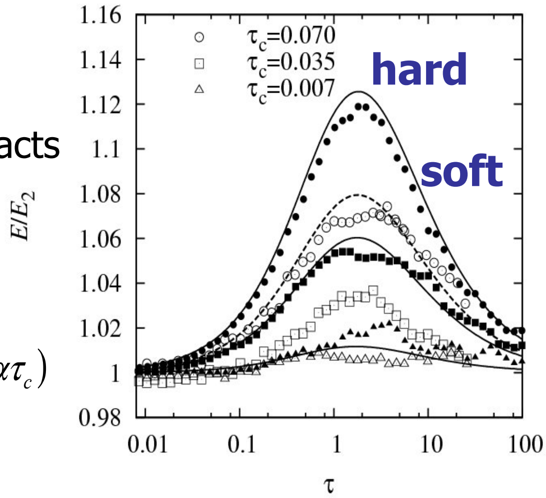
Freely cooling system (HCS->TC-HCS)



Multi-particle contacts

- Contact duration t_c
- Higher density
- Multiple, static contacts
- Smaller dissipation

$$I \to I \exp\left(-\alpha \frac{t_c}{t_n}\right) = I \exp(-\alpha \tau_c)$$



Static vs. dynamic another order parameter?

TC model allows to define

- "potential" energy
- "static" contacts

beyond the limits of hard sphere model validity

$$\tau_c := \frac{t_c}{t_n} > 1: \quad \text{static}$$

$$\tau_c := \frac{t_c}{t_n} < 1: \quad \text{collisional}$$

$+\,$dynamic

Biaxial box element test

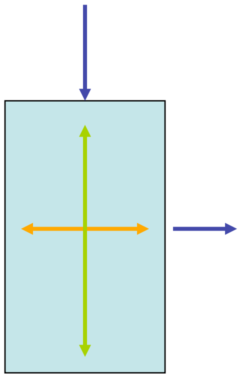
- Top wall: strain controlled

$$z(t) = z_f + \frac{z_0 - z_f}{2}(1 + \cos \omega t)$$

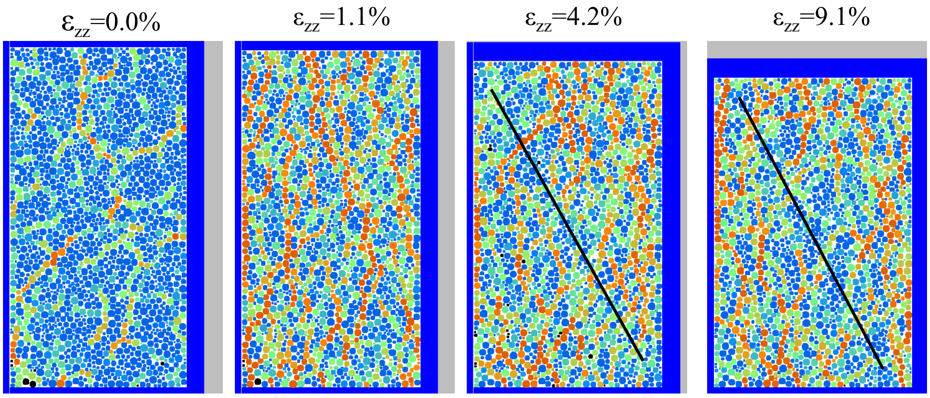
- Right wall: stress controlled

$$p = \text{const.}$$

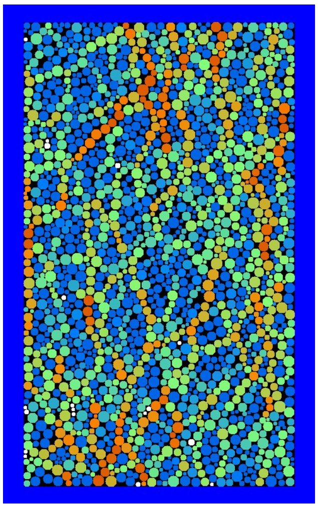
- Evolution with time … ?



Element test simulations



Bi-axial box (stress chains)



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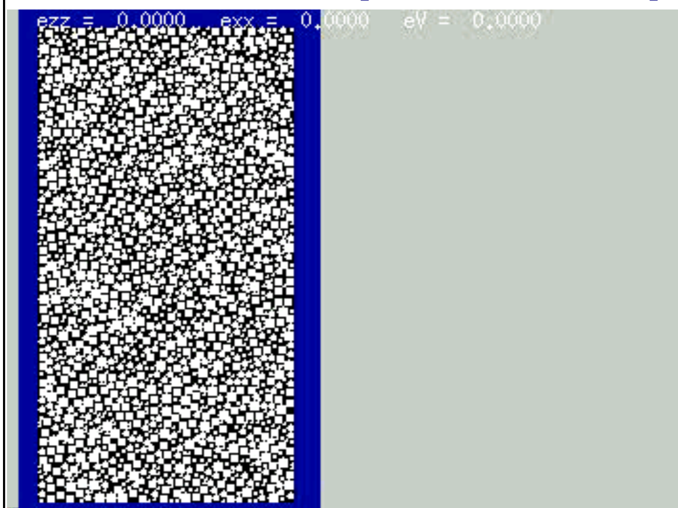
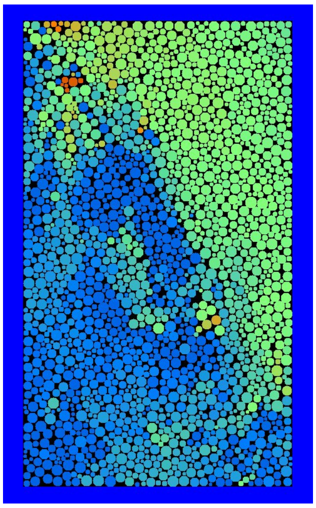
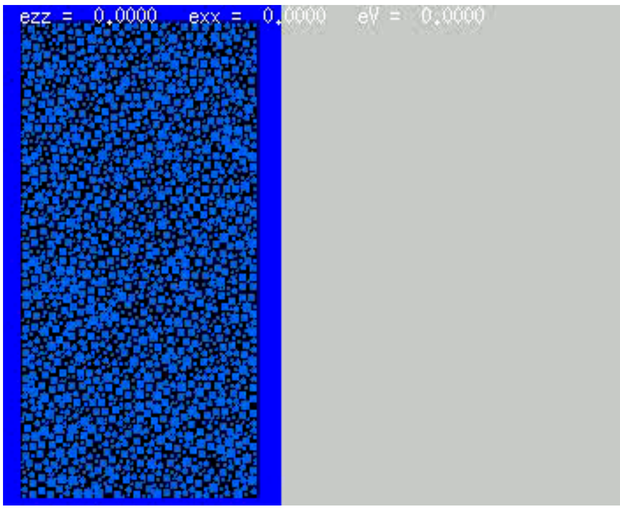


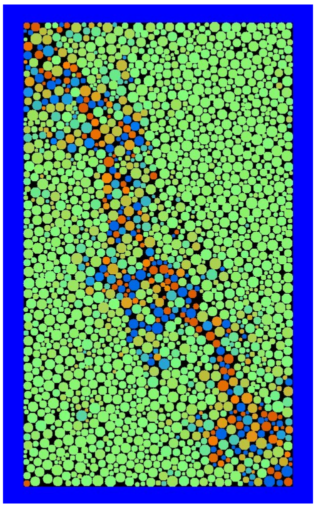
Bi-axial box (kinetic energy)



Bi-axial box (kinetic energy)



Bi-axial box (rotations)



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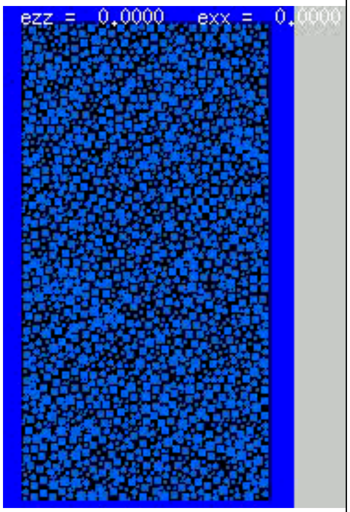
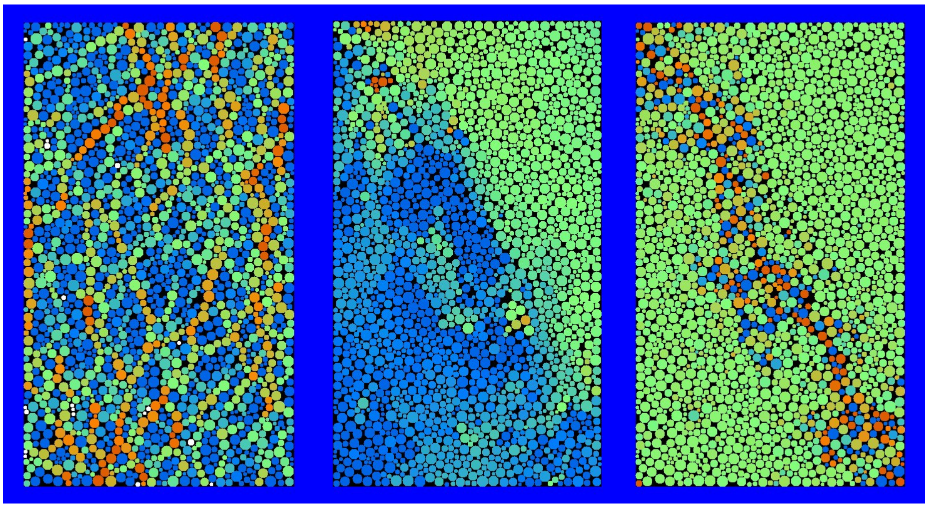
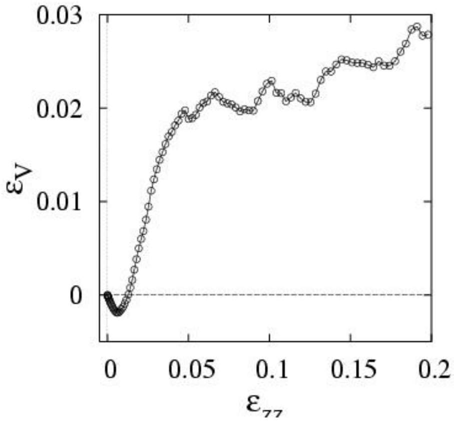


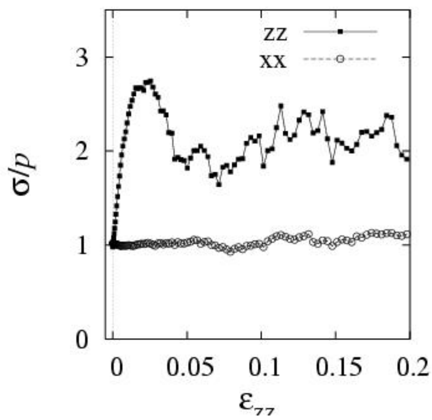
Multiple micro-mechanisms



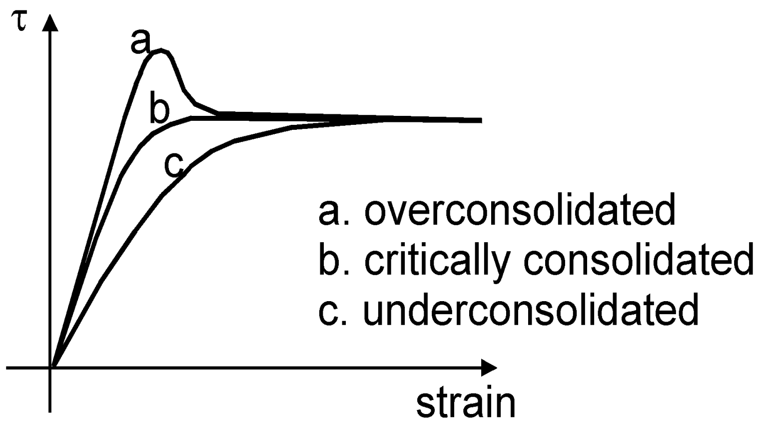
inhomogeneity & anisotropy, instabilities & structures, rotations

Bi-axial compression with p_x=const.

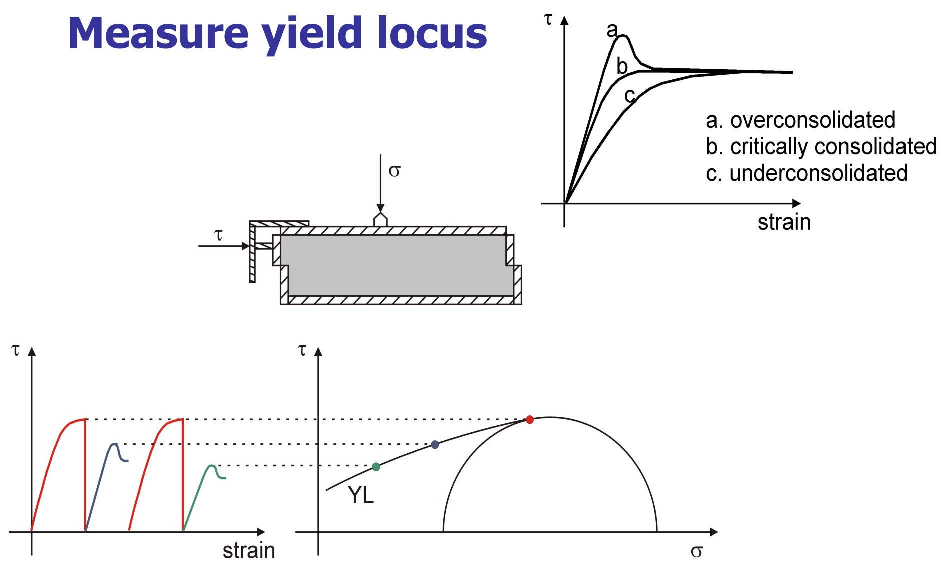




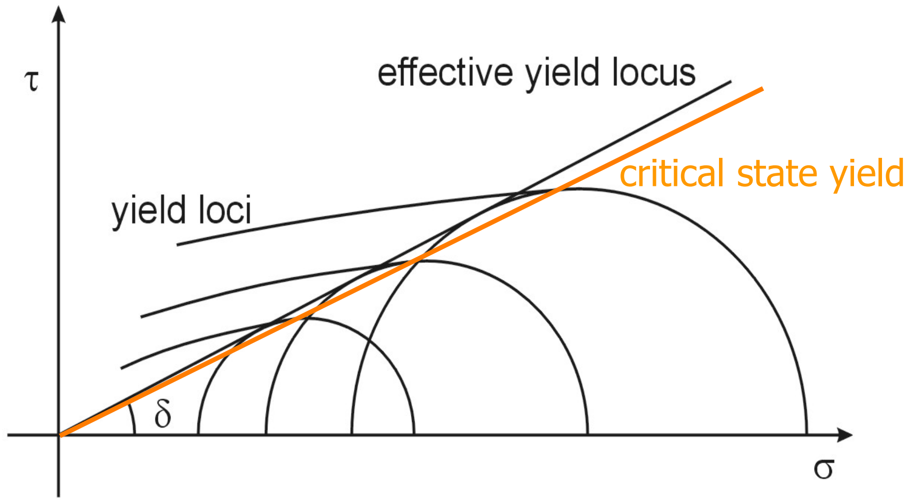
Microscopic interpretation: memory?



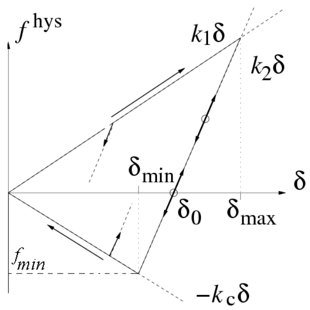
Measure yield locus

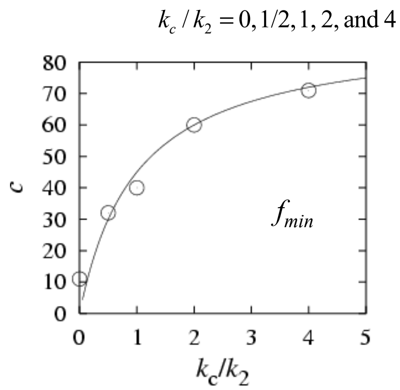


Yield loci



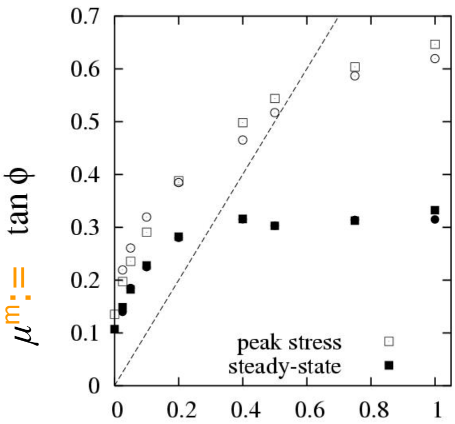
Micro-macro for cohesion





micro adhesion: f_{min} macro cohesion $c = c_0 \frac{1 - k_1/k_2}{1 + k_2/k_c}$

Micro-macro for friction



prepared μ=0

prepared μ=0.5

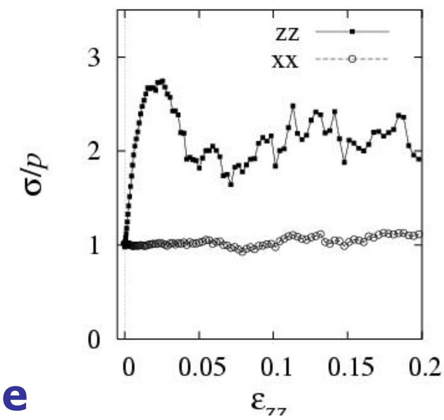
steady-state shear

micro contact-friction μ macro friction-angle μ^m

NOTE: each point = 5-10 simulations

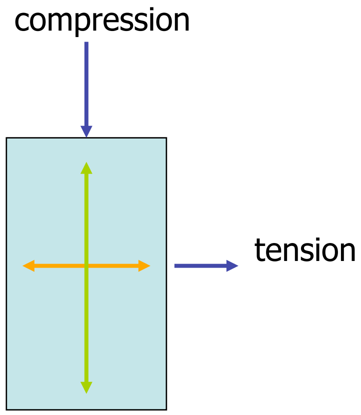
What is relevant?

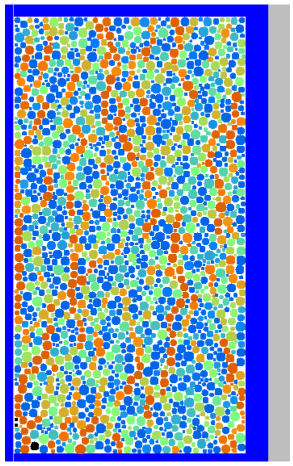
1 – critical state
2 – anisotropy …



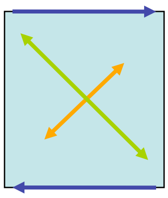
How to find a simple constitutive model?

Micro-macro for anisotropy – rheology





Anisotropy ⇔ Shear ?

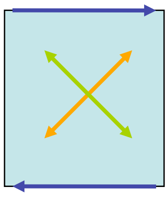


- Simple shear

$$\varepsilon = \begin{pmatrix} 0 & 2\varepsilon_s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_s \\ -\varepsilon_s & 0 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_s \\ \varepsilon_s & 0 \end{pmatrix}$$

Rotation + symmetric shear

Anisotropy ⇔ Shear ?

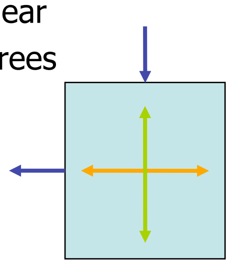


- Simple shear

$$\varepsilon = \begin{pmatrix} 0 & 2\varepsilon_s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_s \\ -\varepsilon_s & 0 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon_s \\ \varepsilon_s & 0 \end{pmatrix}$$

Rotation + symmetric shear

- Rotate symmetric shear tensor by 45 degrees

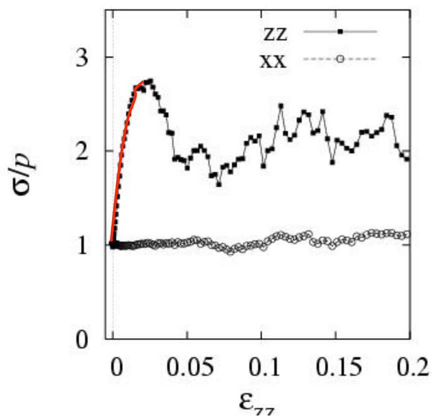


$$R_{45} \cdot \begin{pmatrix} 0 & \varepsilon_s \\ \varepsilon_s & 0 \end{pmatrix} \cdot R_{45}^T = \begin{pmatrix} \varepsilon_s & 0 \\ 0 & -\varepsilon_s \end{pmatrix}$$

- Biaxial “shear”: compression+extension

An-isotropy

in stress



An-isotropy (Stress)

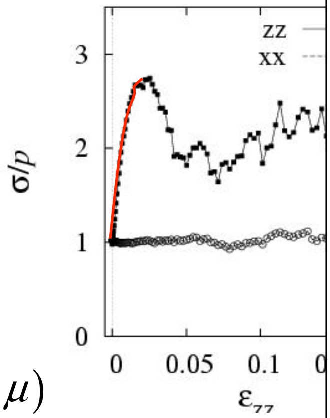
- Stress: Isotropic: $\mathrm{tr}\,\sigma$, and deviatoric: $\mathrm{dev}\,\sigma = \sigma_{zz}-\sigma_{xx}$
  - Minimal eigenvalue: σ_{xx}
  - Maximal eigenvalue: σ_{zz}
- Dev. Stress fraction $s_D = \mathrm{dev}\,\sigma / \mathrm{tr}\,\sigma$

$$\frac{\partial}{\partial \varepsilon_D} s_D = \beta_s \left(s_{\max} - s_D \right)$$

- Exponential approach to peak

$$1 - s_D / s_{\max} = \exp\left(-\beta_s \varepsilon_D\right)$$

$\beta_s\left(\rho, p, \mu\right)$

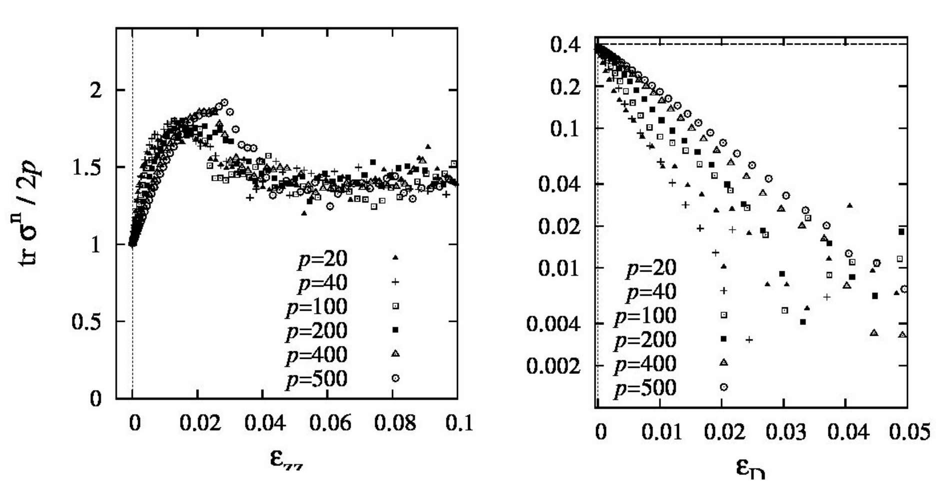


An-isotropy (Stress)

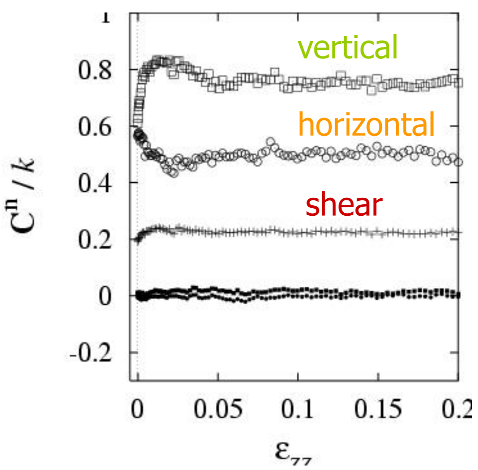
$$\frac{\partial}{\partial \varepsilon_D} s_D = \beta_s \left(s_{\max} - s_D \right)$$

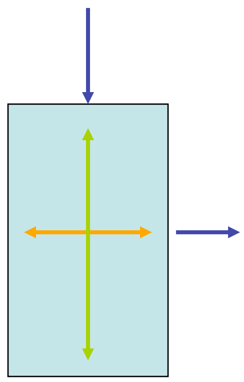
Stress (homog.)

$$1 - s_D / s_{max} = \exp(-\beta_s \varepsilon_D)$$



Stiffness tensor





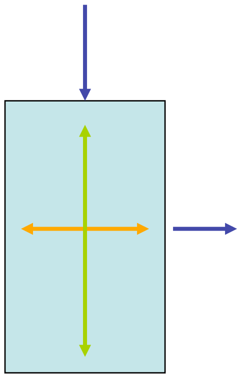
Different moduli:

- against shear C_2
- perpendicular C_1
- *one* shear modulus

An-isotropy (Structure)

- Structure changes with deformation
- Different stiffness:
  - More stiffness against shear C_2
  - Less stiffness perpendicular C_1
- One (only?) shear modulus
- Anisotropy $A = C_2 - C_1$ evolution

$$\frac{\partial}{\partial \varepsilon_D} A = \beta_F \left(A_{\max} - A \right)$$

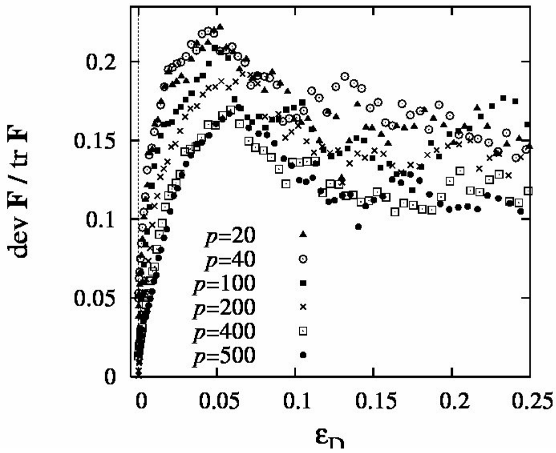


- Exponential approach to maximal anisotropy

… see Calvetti et al. 1997

Fabric

$$\frac{\partial}{\partial \varepsilon_D} A = \beta_F \left(A_{\max} - A \right)$$



An-isotropy (Stress & Structure)

$$\frac{\partial}{\partial \varepsilon_D} s_D = \beta_s \left(s_{\max} - s_D \right)$$

$$\frac{\partial}{\partial \varepsilon_D} A = \beta_F \left(A_{\max} - A \right)$$

An-isotropy (Stress & Structure)

Modulus

Friction

$$\frac{\partial}{\partial \varepsilon_D} s_D = \beta_s \left(s_{\max} - s_D \right)$$

$$\frac{\partial}{\partial \varepsilon_D} A = \beta_F \left(A_{\max} - A \right)$$

Constitutive model
scalar! (in the biaxial box eigen-system)

Isotropic stress $\delta p = \delta\sigma_V = 2B\varepsilon_V + AS\,d\gamma$

Deviatoric stress $\delta\tau = \delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma$

Anisotropy $\delta A = \beta_A\left(A^{\max} - A\right)\left|d\gamma\right|$

stress-isotropy $S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model – isotropic mat.
scalar! (in the biaxial box eigen-system)

Isotropic stress $\delta\sigma_V = 2B\varepsilon_V$

Deviatoric stress $\delta\tau = 2GS\,d\gamma$

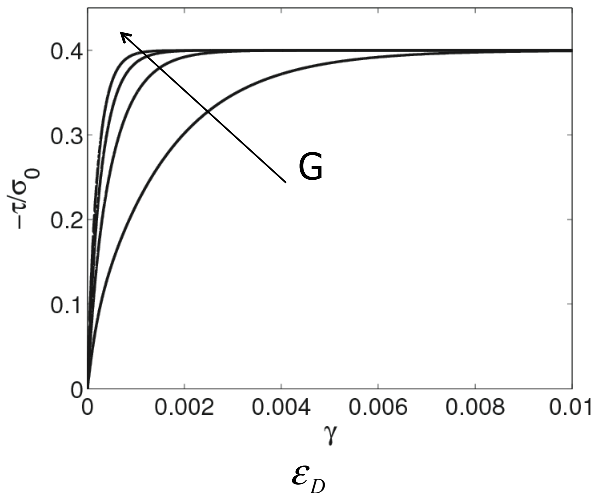
Anisotropy $A = 0$

stress-isotropy $S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model – scalar



Constitutive model
various deformation modes

Mode 0: Isotropic $d\gamma = 0$

Mode 1: Uni-axial

Mode 2: Deviatoric $\varepsilon_V = 0$

Mode 3: Bi-axial (side-stress controlled)

Mode 4: Bi-axial (isobaric, p-contolled)

Constitutive model – isotropic (mode 0)
scalar! (in the biaxial box eigen-system)

Isotropic stress $\delta\sigma_V = 2B\varepsilon_V$

Deviatoric stress $\delta\tau = A\varepsilon_V$

Anisotropy $\delta A = 0$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model – isobaric (mode 4)
scalar! (in the biaxial box eigen-system)

Isotropic stress $0 = 2B\varepsilon_V + AS\,d\gamma$

Deviatoric stress $\delta\tau = \delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma$

Anisotropy $\delta A = \beta_A\left(A^{\max} - A\right)\left|d\gamma\right|$

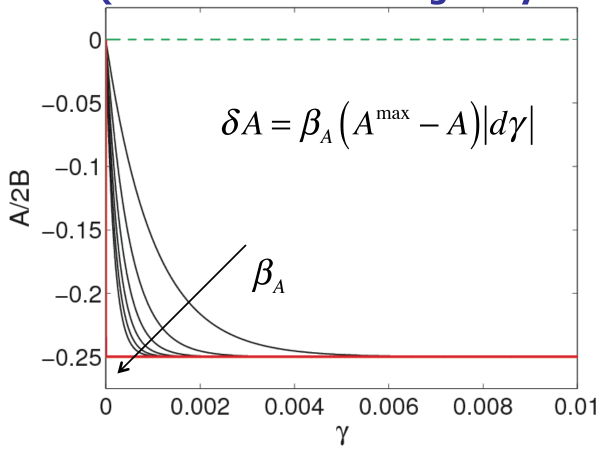
stress-isotropy $S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

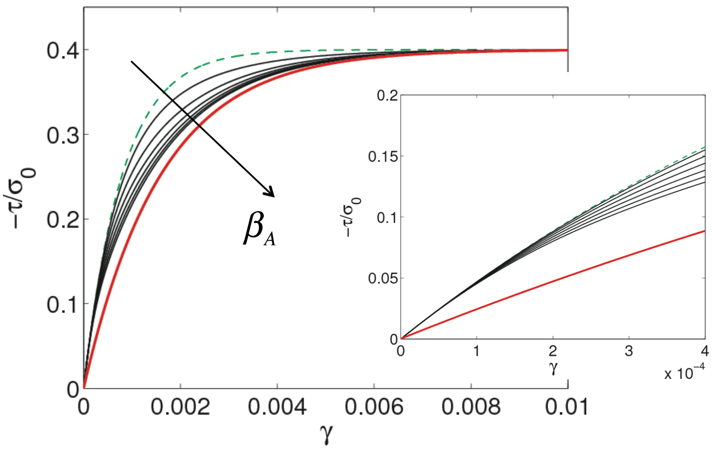
B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model – scalar

(in the biaxial box eigen-system)



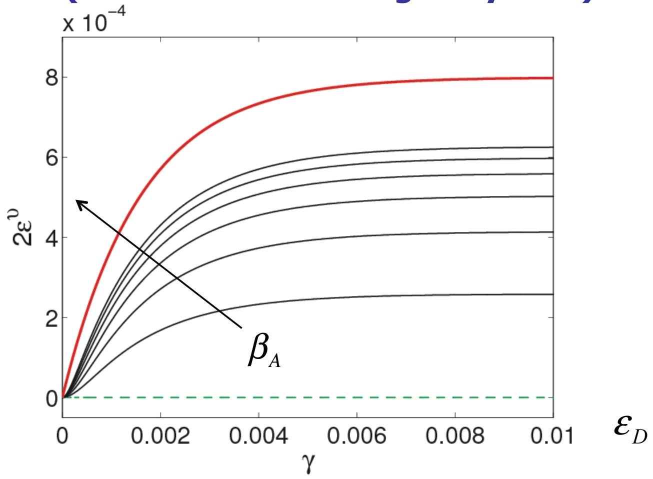
Constitutive model – scalar



ε_D

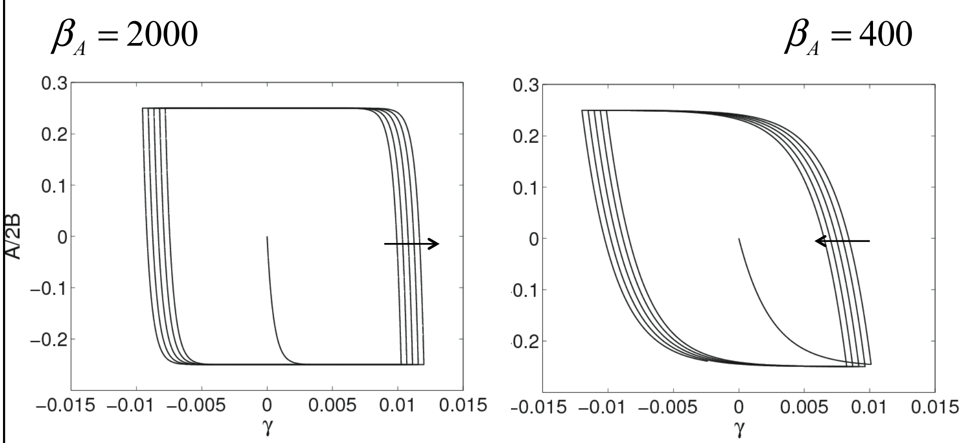
Constitutive model – scalar

(in the biaxial box eigen-system)



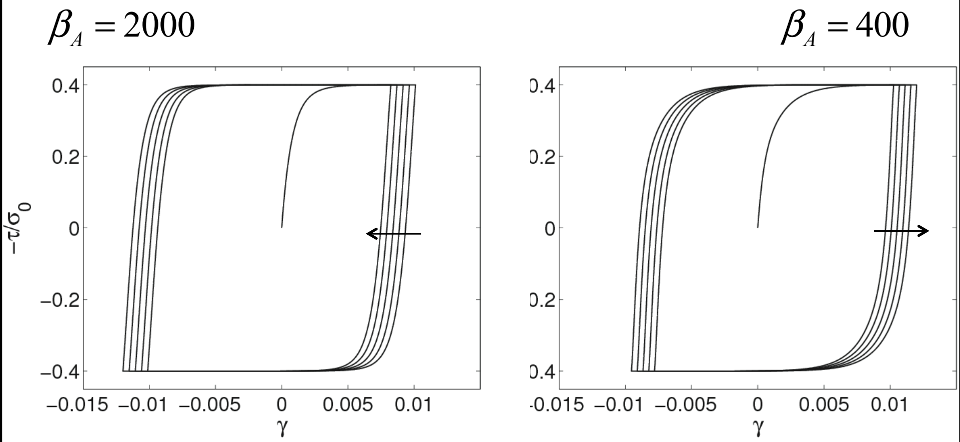
Constitutive model – cyclic loading

(in the biaxial box eigen-system)

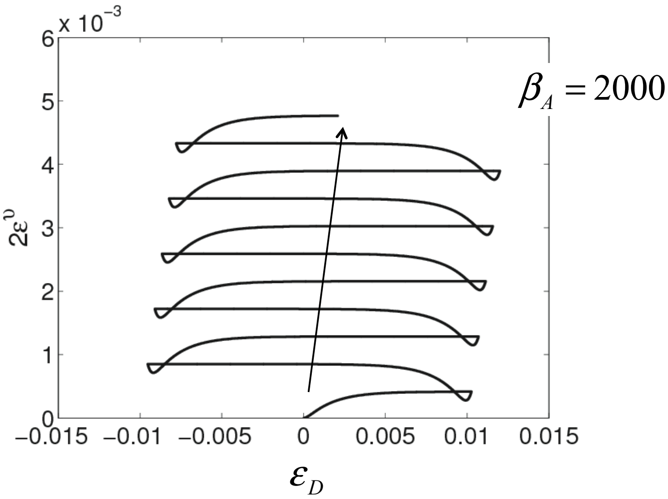


Constitutive model – cyclic loading

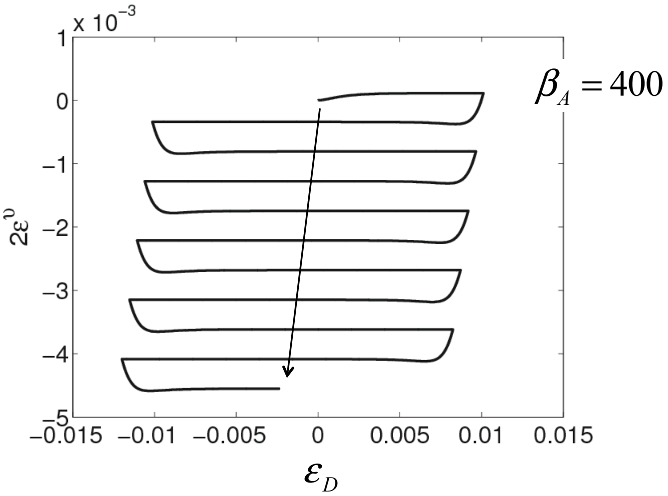
(in the biaxial box eigen-system)



Constitutive model – scalar: dilatancy



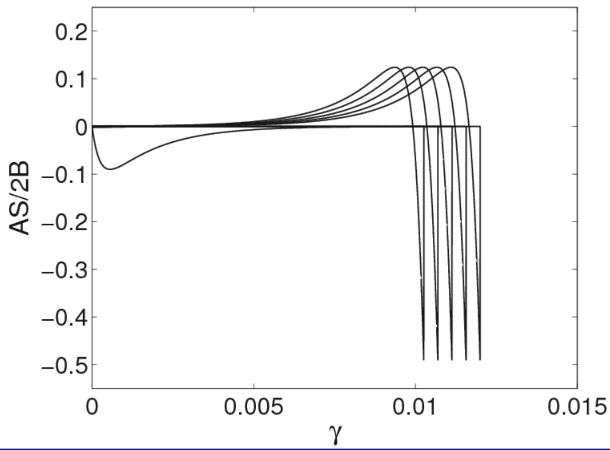
Constitutive model – scalar: contractancy



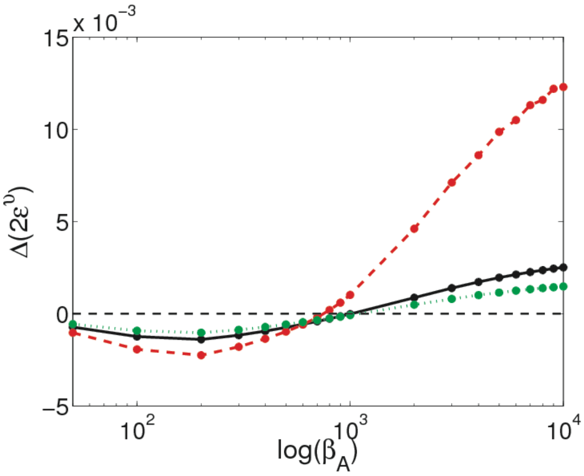
Constitutive model – scalar

(in the biaxial box eigen-system)

$$0 = 2B\varepsilon_V + AS\varepsilon_D$$



Constitutive model – anisotropy rate



Constitutive model
scalar! (in the biaxial box eigen-system)

Isotropic stress $$0 = 2B\varepsilon_V + AS\,d\gamma$$

Deviatoric stress $$\delta\tau = \delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma$$

Anisotropy $$\delta A = \beta_A \left(A^{\max} - A\right)\left|d\gamma\right|$$

stress-isotropy $$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model – scalar

(in the biaxial box eigen-system)

Bulk modulus B:
compression leads to pressure

Shear modulus G:
shear strain leads to shear stress

Anisotropy:
shear strain leads to pressure
compression leads to shear-stress

Cross-coupling of isotropic and deviatoric parts

Constitutive model – scalar

(in the biaxial box eigen-system)

Anisotropy:

Strain-controlled:
shear strain leads to pressure
compression leads to shear-stress

Stress-controlled:
shear strain leads to dilatancy/compactancy
compression leads to shear-deformation

Time-scales

- Contact duration tc
- Inverse shear rate
- Time between collisions
- Inverse dissipation rate
- stress-change?
- Anisotropy-change?

Constitutive model
scalar! (in the biaxial box eigen-system)

Isotropic stress $\delta p = \delta\sigma_V = 2B\varepsilon_V + AS\,d\gamma$

Deviatoric stress $\delta\tau = \delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma$

Anisotropy $\delta A = \beta_A\left(A^{\max} - A\right)\left|d\gamma\right|$

stress-isotropy $S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$

Isotropic|deviatoric strain increment $\varepsilon_V \mid d\gamma$

B … Bulk-, *G* … Shear-, *A* … Anisotropy-Modulus

Constitutive model scalar! … but where is the time-scale?

$$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

Isotropic stress $\delta p = \delta\sigma_V = 2B\varepsilon_V + AS\,d\gamma$

Deviatoric stress $\delta\tau = \delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma$

Anisotropy $\delta A = \beta_A \left(A^{\max} - A\right)\left|d\gamma\right|$

$\varepsilon_V \mid d\gamma$

Constitutive model scalar! … but where is the time-scale?

$$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

Isotropic stress $\delta p = 2B\varepsilon_V + AS\,d\gamma - 1/\tau_p \, p\,dt$

Deviatoric stress $\delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma - 1/\tau_D \, \sigma_D\,dt$

Anisotropy $\delta A = \beta_A \left(A^{\max} - A\right)\left|d\gamma\right| - 1/\tau_A \, A\,dt$

$\varepsilon_V \mid d\gamma$

Constitutive model scalar! ... but where is the time-scale?

$$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

Isotropic stress $\delta p = 2B\varepsilon_V + AS\,d\gamma - \frac{1}{\tau_p} p\,dt$

Deviatoric stress $\delta\sigma_D = A\varepsilon_V + 2GS\,d\gamma - \frac{1}{\tau_D}\sigma_D\,dt$

~~Anisotropy~~ ~~$\delta A = \beta_A\left(A^{\max} - A\right)|d\gamma| - \frac{1}{\tau_A} A\,dt$~~

Isotropy?

$\varepsilon_V \mid d\gamma$

Constitutive model scalar! ... but where is the time-scale?

Isotropic stress $\delta p = 2B\varepsilon_V - \frac{1}{\tau_p} p\,dt$

Deviatoric stress $\delta\sigma_D = 2G\,d\gamma - f\sigma_D\,dt$

$\varepsilon_V \mid d\gamma$

Constitutive model
scalar! … but where is the time-scale?

Isotropic stress $\delta p = 2B\varepsilon_V - \frac{1}{\tau_p} p\, dt$

Deviatoric stress $\delta\sigma_D = 2G\, d\gamma - f\sigma_D\, dt$

fluidity (Nguyen et al. 2011)
… with an evolution equation by its own …

$f \propto \dot{\gamma}$

Steady (critical) state: $2\frac{G}{f}\dot{\gamma} = \sigma_D^{\max} = \alpha G$

$\varepsilon_V \mid d\gamma$

Constitutive model
scalar! … but where is the time-scale?

$S = 1 - \frac{\sigma_D}{\sigma_D^{\max}} = 1 - \frac{s_D}{s_D^{\max}}$

Isotropic stress $\delta p = 2B\varepsilon_V + AS\, d\gamma - \frac{1}{\tau_p} p\, dt$

Deviatoric stress $\delta\sigma_D = A\varepsilon_V + 2GS\, d\gamma - \frac{1}{\tau_D}\sigma_D\, dt$

~~Anisotropy~~ ~~$\delta A = \beta_A\left(A^{\max} - A\right)|d\gamma| - \frac{1}{\tau_A} A\, dt$~~
Isotropy?

Granular Solid Hydrodynamics
GSH-type formulation (M. Liu 2003-2011)

$\varepsilon_V \mid d\gamma$

Constitutive model scalar! … but where is the time-scale?

$$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

Isotropic stress $\delta p = 2B\varepsilon_V - \frac{1}{\tau_p} p\,dt$

Deviatoric stress $\delta\sigma_D = 2G\,d\gamma - \frac{1}{\tau_D}\sigma_D\,dt$

different relaxation times for *p* and *D* $\frac{1}{\tau} \propto T_g \propto \dot{\gamma}$

Granular Solid Hydrodynamics
GSH-type formulation (M. Liu 2003-2011)

$\varepsilon_V \mid d\gamma$

Constitutive model isotropic! …
… in the critical state!

Deviatoric stress (Luding) $0 = 2GS\,d\gamma$

Deviatoric stress (GSH) $0 = 2G\,d\gamma - \frac{1}{\tau_D}\sigma_D\,dt$

relaxation rate

viscosity

GSH-type formulation (M. Liu 2003-2011) $\varepsilon_V \mid d\gamma$

Constitutive model isotropic! …
… in the critical state!

Deviatoric stress (Luding) $$0 = 2G\left(1 - {s_D}/{s_D^{\max}}\right) d\gamma$$

Deviatoric stress (GSH) $$0 = 2G\,d\gamma - {1}/{\tau_D}\,\sigma_D^{\max}\,dt$$

relaxation rate $$1/\tau_D = \frac{2G}{\sigma_D^{\max}}\dot{\gamma} = \frac{2G/p}{s_D^{\max}}\dot{\gamma}$$

viscosity $$\eta = \sigma_D^{\max}/\dot{\gamma} = 2G\tau_D$$

GSH-type formulation (M. Liu 2003-2011) $\varepsilon_V \mid d\gamma$

Time-scales

- Contact duration tc
- inverse shear rate
- Time between collisions
- inverse dissipation rate
- inverse isotropic pressure-change rate
- inverse anisotropic stress-change rate
- inv. Anisotropy-change rate
- Non-co-linearity relaxation?

Interaction of time-scales?

Constitutive model

$$S = 1 - \sigma_D / \sigma_D^{\max} = 1 - s_D / s_D^{\max}$$

scalar! … but where is the time-scale?

How to measure, e.g., time-scale τ_D

Deviatoric stress $$\delta\sigma_D = 2G\,d\gamma - \frac{1}{\tau_D}\sigma_D\,dt$$

stop! $$\dot{\sigma}_D = -\frac{1}{\tau_D}\sigma_D$$ $$\frac{1}{\tau_D(t)} \propto T_g$$

$$\dot{T}_g = -I$$

$\varepsilon_V \mid d\gamma$

Constitutive model

scalar! … but where is the time-scale?

How to measure, e.g., non-colinearity ϕ_σ

Relaxation model: $$\delta\phi_\sigma = \tfrac{1}{2}\,d\gamma_s - \frac{1}{\tau_\phi}\left(\phi_\sigma - \phi_\varepsilon\right)dt$$

in general non-colinear (also for A)!

Note the difference between γ_s and γ

$\varepsilon_V \mid d\gamma$

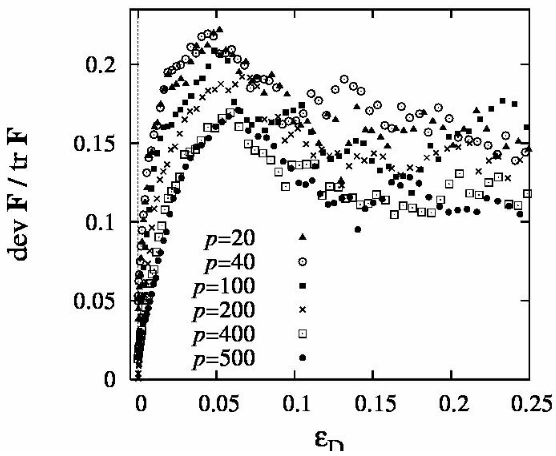
How to simplify?

1 – critical state
2 – anisotropy …

Minimal constitutive model:
B, G, $\mu = s_D^m$, A^m, β_A

Fabric

$$\frac{\partial}{\partial \varepsilon_D} A = \beta_F \left(A_{max} - A \right)$$



Open Issues: Evolution for arbitrary orientation

Constitutive model with rotations

$$\delta\boldsymbol{\sigma} = E\varepsilon_V \left(\mathbf{1} + G^* \frac{\varepsilon_D}{\varepsilon_V} \hat{\mathbf{D}}_\sigma \right) + F_D \left(S\varepsilon^* \hat{\mathbf{D}}_{F+45^o} + R\varepsilon^{**} \hat{\mathbf{D}}_{F+90^o} \right) + \delta\boldsymbol{\sigma}_A$$

particle eigen-rotations: $\theta_i^* = \theta_i - \theta_i^c$

sliding component: $\varepsilon^* = a_1 \delta\theta_1^* + a_2 \delta\theta_2^*$

rolling component: $\varepsilon^{**} = a_1 \delta\theta_1^* - a_2 \delta\theta_2^*$

with: sliding stiffness: S and rolling stiffness: R

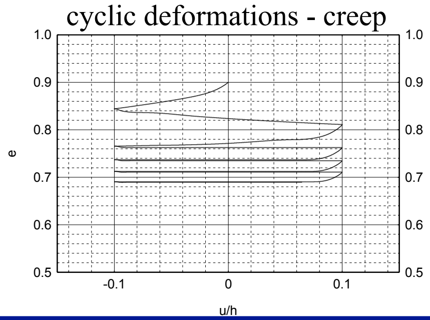
unit-deviator operator: $\hat{\mathbf{D}}_{F+45^o} = \mathbf{R}^T \left(\phi_F + \tfrac{\pi}{4}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}\left(\phi_F + \tfrac{\pi}{4}\right)$

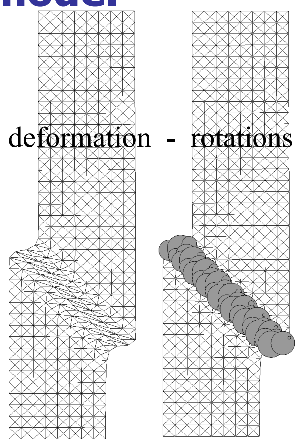
Open Issues: Evolution with rotations

Implementation in FEM model

+ successful tool – few parameters
- microscopic foundations ?
- extensions & parameter identification

cyclic deformations - creep





Continuum Theory

Thank you! ☺

Questions?