MANCHESTER 1824

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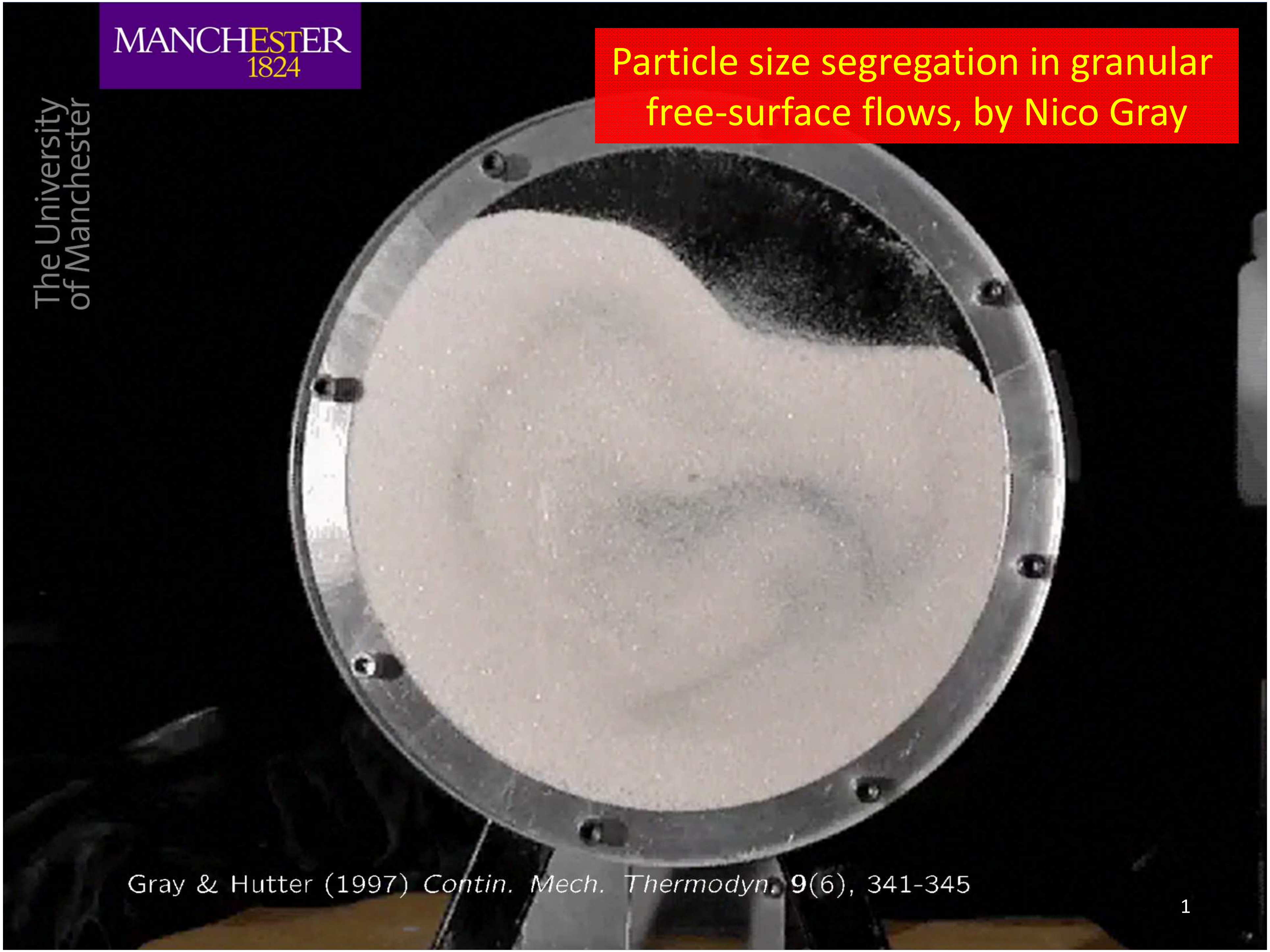
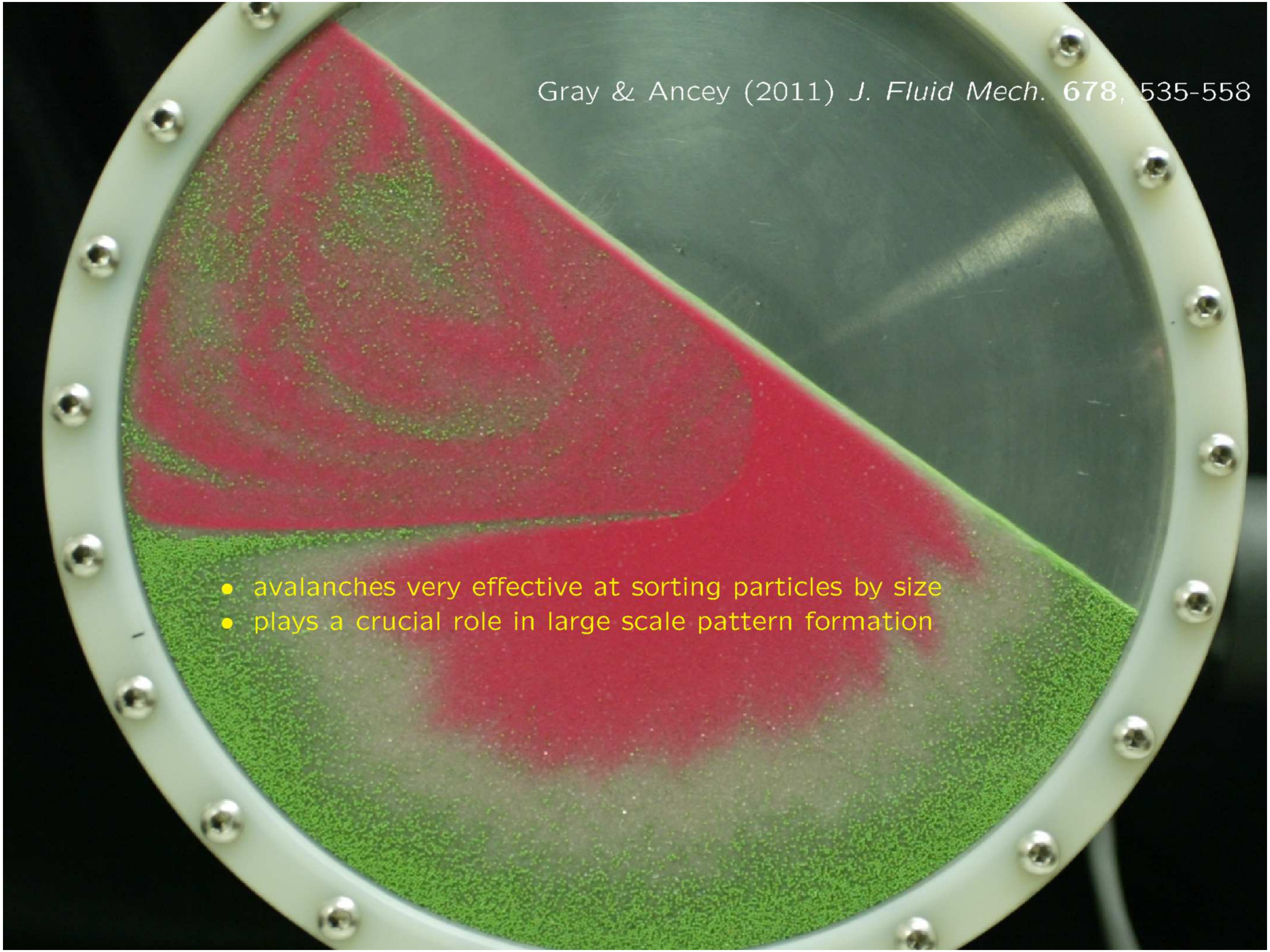
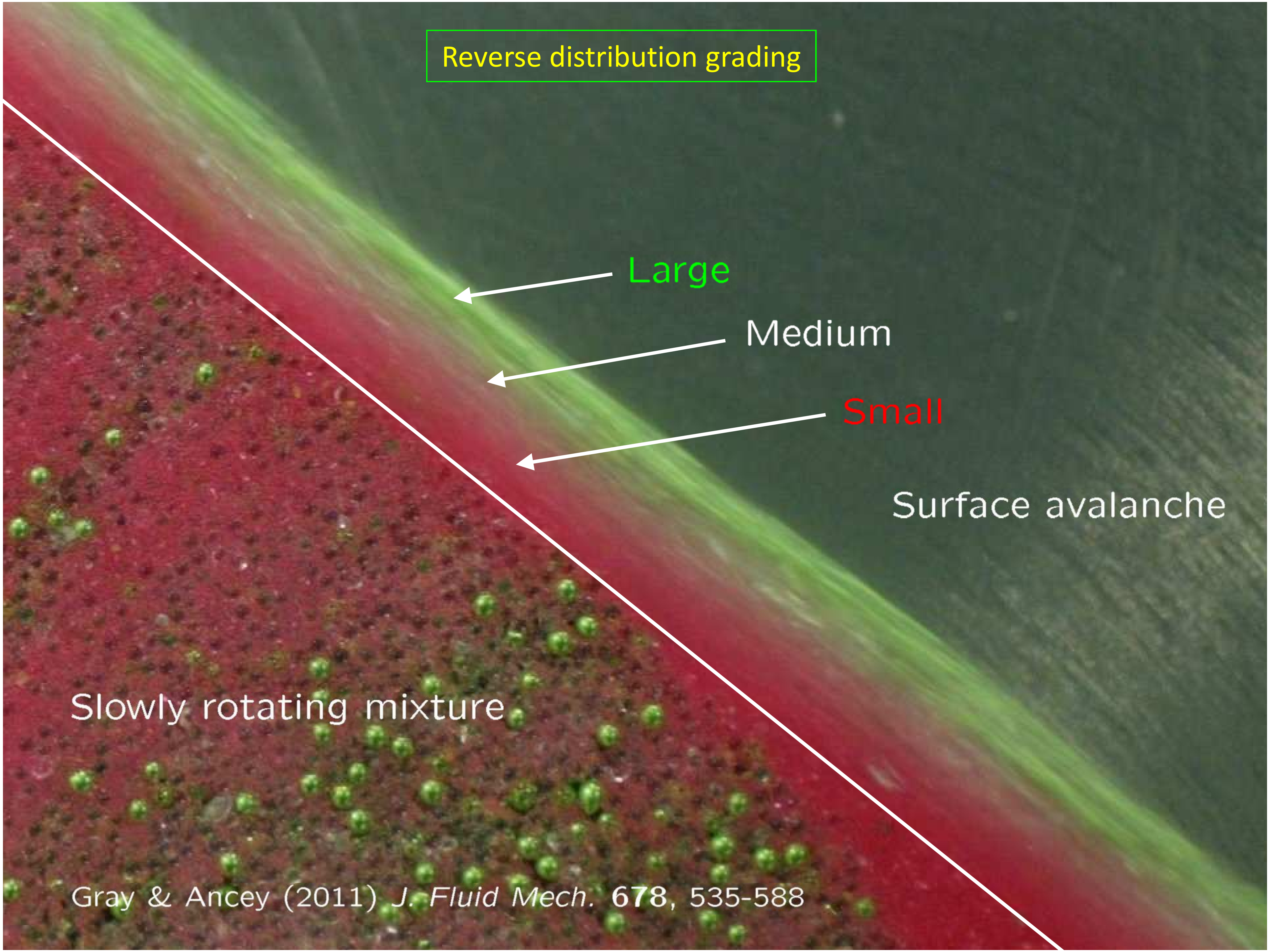
# Particle size segregation in granular free-surface flows, by Nico Gray

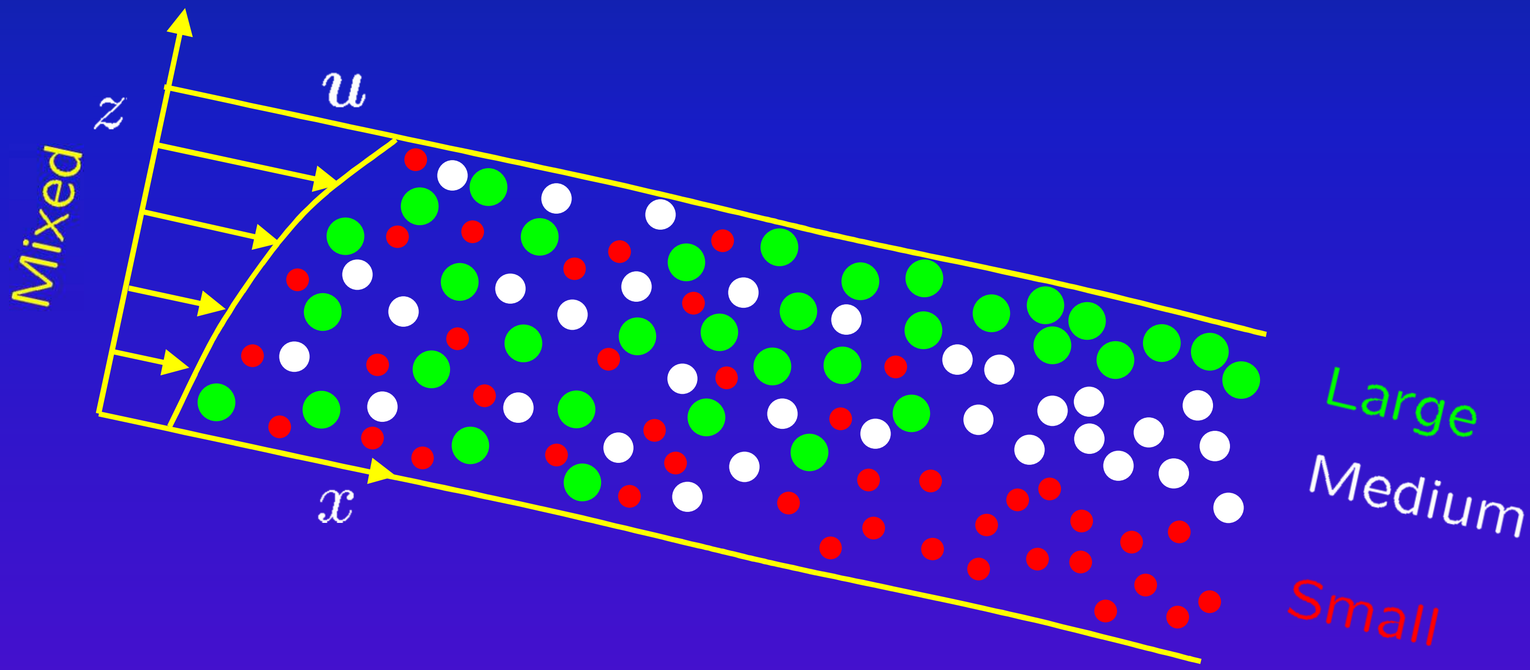


Gray & Hutter (1997) *Contin. Mech. Thermodyn.* **9**(6), 341-345


Gray & Ancey (2011) J. Fluid Mech. 678, 535-558
• avalanches very effective at sorting particles by size
• plays a crucial role in large scale pattern formation


Reverse distribution grading
Large
Medium
Small
Surface avalanche
Slowly rotating mixture
Gray & Ancey (2011) J. Fluid Mech. 678, 535-588

# Particle-size segregation and diffusive-remixing




- kinetic sieving is the key segregation mechanism
  - small particles more likely to fall down into gaps
  - and then force large particles up
  - to create inversely graded layers
- diffusive-remixing competes against this

## Mixture framework

- the volume fraction $\phi^\nu$ of constituent $\nu$, per unit volume of mixture, lies in the range

$$0 \leq \phi^\nu \leq 1.$$

and their sum

$$\sum_{\forall \nu} \phi^\nu = 1.$$

- In standard mixture theory the partial and intrinsic density, stress, pressure and velocity fields satisfy

$$\rho^\nu = \phi^\nu \rho^{\nu *}, \quad \boldsymbol{\sigma}^\nu = \phi^\nu \boldsymbol{\sigma}^{\nu *}, \quad p^\nu = \phi^\nu p^{\nu *}, \quad \boldsymbol{u}^\nu = \boldsymbol{u}^{\nu *}$$

- The bulk density and pressure are defined as

$$\rho = \sum_{\forall \nu} \rho^\nu, \quad p = \sum_{\forall \nu} p^\nu.$$

## Mass and momentum balances for each constituent

- Each constituent satisfies individual mass

$$\frac{\partial \rho^\nu}{\partial t} + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu) = 0,$$

and momentum balances

$$\frac{\partial}{\partial t}(\rho^\nu \boldsymbol{u}^\nu) + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu \otimes \boldsymbol{u}^\nu) = \nabla \cdot \boldsymbol{\sigma}^\nu + \rho^\nu \boldsymbol{g} + \boldsymbol{\beta}^\nu,$$

where $\otimes$ is the dyadic product and $\boldsymbol{g}$ is the gravitational acceleration vector.

- The interaction force $\boldsymbol{\beta}^\nu$ is the force exerted on phase $\nu$ by all the other constituents. Their sum

$$\sum_{\forall \nu} \boldsymbol{\beta}^\nu = \mathbf{0}.$$

## Bulk lithostatic pressure

- Particles have the same intrinsic density $\rho^{\nu *} = \rho_0$
- Acceleration terms are negligible in the $z$ direction.
- The normal momentum balances sum to

$$\frac{\partial p}{\partial z} = -\rho g \cos\zeta,$$

where $g$ is the gravitational acceleration.

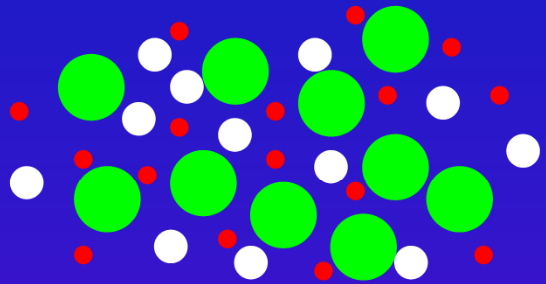
- Since $\rho$ is constant and the free-surface $z = s(x,t)$ is traction free, this can be integrated to show that the pressure is lithostatic

$$p = \rho g (s - z) \cos\zeta.$$

- Compatible with existing avalanche models

## Non-standard partial/intrinsic pressure relation

- As small particles percolate downwards, they support less of the overburden pressure $\Rightarrow$ large grains support more of the load.



- suggests a partial/intrinsic pressure relation that differs from standard mixture theory

$$p^\nu = f^\nu p,$$

where $f^\nu \neq \phi^\nu$ determines the proportion of the pressure carried by each constituent. Their sum

$$\sum_{\forall \nu} f^\nu = 1.$$

# Interaction drag

- Interaction drag consists of three terms

$$\boldsymbol{\beta}^\nu = p\nabla f^\nu - \rho^\nu c(\boldsymbol{u}^\nu - \boldsymbol{u}) - \rho d \nabla \phi^\nu,$$

$c$ is the coefficient of inter-particle drag,
$d$ is the coefficient of diffusive remixing

- barycentric (bulk) velocity defined by

$$\rho \boldsymbol{u} = \sum_{\forall \nu} \rho^\nu \boldsymbol{u}^\nu.$$

- The constituent velocities are assumed equal to the bulk velocity in the down- and cross-slope directions

$$u^\nu = u, \qquad v^\nu = v.$$

## Momentum balance of each constituent in the normal direction

- Assuming normal accelerations are negligible

$$\phi^\nu w^\nu = \phi^\nu w + (f^\nu - \phi^\nu)(g/c)\cos\zeta - (d/c)\frac{\partial \phi^\nu}{\partial z},$$

$$\begin{array}{ll} f^\nu - \phi^\nu > 0 & \text{particles rise} \\ f^\nu - \phi^\nu = 0 & \text{no relative motion} \\ f^\nu - \phi^\nu < 0 & \text{particles percolate downwards} \end{array}$$

- When any class of particles are in a pure phase they must carry all of the load

$$f^\nu = 1, \quad \text{when} \quad \phi^\nu = 1,$$

- When there are no particles of that phase, they cannot carry any of the load

$$f^\nu = 0, \quad \text{when} \quad \phi^\nu = 0.$$

## Additive decomposition of the perturbations

- If any two constituents are found in isolation, the form of $f^\nu$ must reduce to bidisperse case, e.g.

$$f^l = \phi^l + B_{ls}\phi^l\phi^s,$$

- This suggests an additive decomposition

$$f^\nu = \phi^\nu + \sum_{\forall\mu} B_{\nu\mu}\phi^\nu\phi^\mu,$$

- $B_{(\nu\nu)} = 0, \quad \forall\nu,$ no perturbations exerted by any constituent on itself

- $B_{\nu\mu} = -B_{\mu\nu}, \quad \forall\nu \neq \mu,$ $\nu$ equal and opposite to $\mu$

- then $\sum f^\nu = 1$ is automatically satisfied.

## The multi-component segregation remixing equation

- Non-dimensionalizing on typical thickness $H$, length $L$ and velocity $U$ implies that

$$w^{\nu} = w + \sum_{\forall \mu} S_{\nu\mu}\phi^{\mu} - D_r \frac{\partial}{\partial z}(\ln \phi^{\nu}),$$

where

$$S_{\nu\mu} = \frac{Lg\cos\zeta}{HUc} B_{\nu\mu}, \qquad D_r = \frac{DL}{H^2 U}.$$

- The non-dimensional segregation remixing equation for phase $\nu$ is therefore

$$\frac{\partial \phi^{\nu}}{\partial t} + \nabla \cdot (\phi^{\nu} \boldsymbol{u}) + \frac{\partial}{\partial z}\left(\sum_{\forall \mu} S_{\nu\mu}\phi^{\nu}\phi^{\mu}\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^{\nu}}{\partial z}\right),$$

## Bi-disperse mixtures

- The multi-component theory yields two equations for the large and small particles

$$\begin{aligned}\frac{\partial \phi^l}{\partial t} + \nabla\cdot(\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right),\\ \frac{\partial \phi^s}{\partial t} + \nabla\cdot(\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s\phi^l) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).\end{aligned}$$

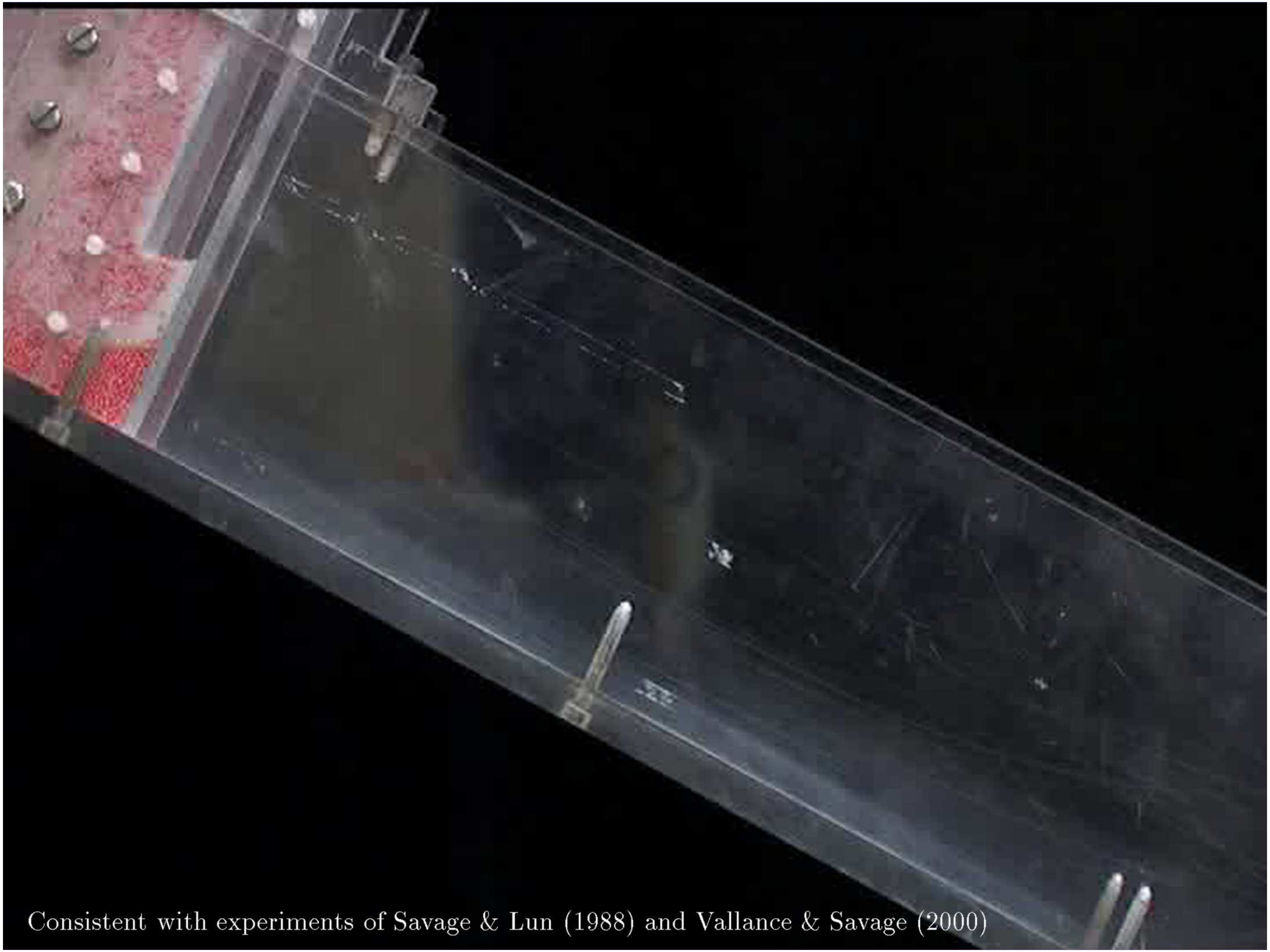
- The summation condition $\sum \phi^\nu = 1$ implies

$$\phi^l + \phi^s = 1,$$

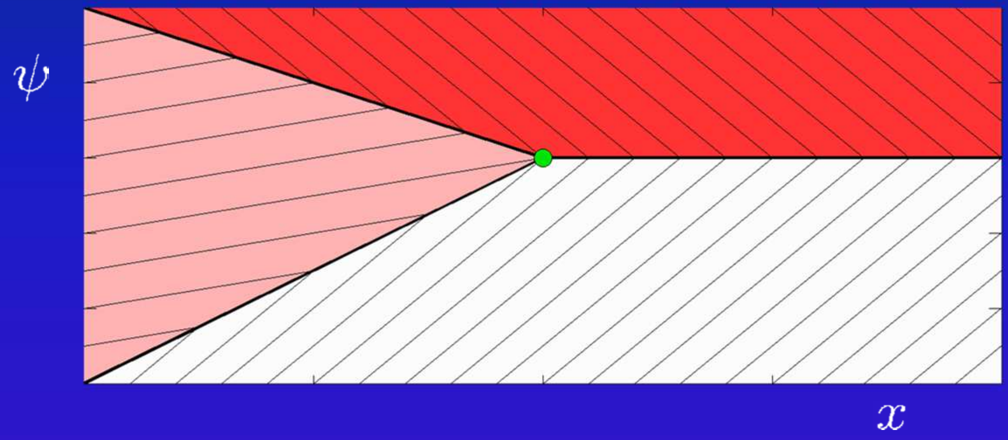
- Large particle concentration can be eliminated

$$\frac{\partial \phi^s}{\partial t} + \nabla\cdot(\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s(1-\phi^s)) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).$$

Gray & Thornton (2005) *Proc. Roy. Soc. A.* **461**, 1447-1473.
Gray & Chugunov (2006) *J. Fluid Mech.* **569**, 365-398.


Consistent with experiments of Savage & Lun (1988) and Vallance & Savage (2000)

## Steady-state concentration shocks in absence of diffusive-remixing




- shock height $s(x)$ satisfies the jump condition

$$\left[\!\left[ \phi u \frac{ds}{dx} + S_{ls}\phi(1-\phi) \right]\!\right] = 0 \quad \Rightarrow \quad u\frac{ds}{dx} = S_{ls}(\phi^+ + \phi^- - 1)$$

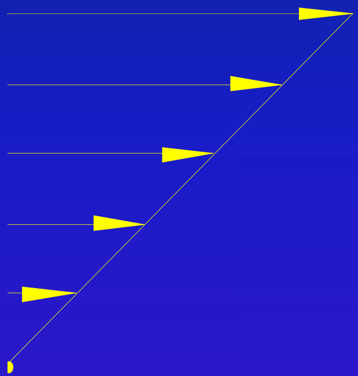
- Using depth-integrated velocity coordinates

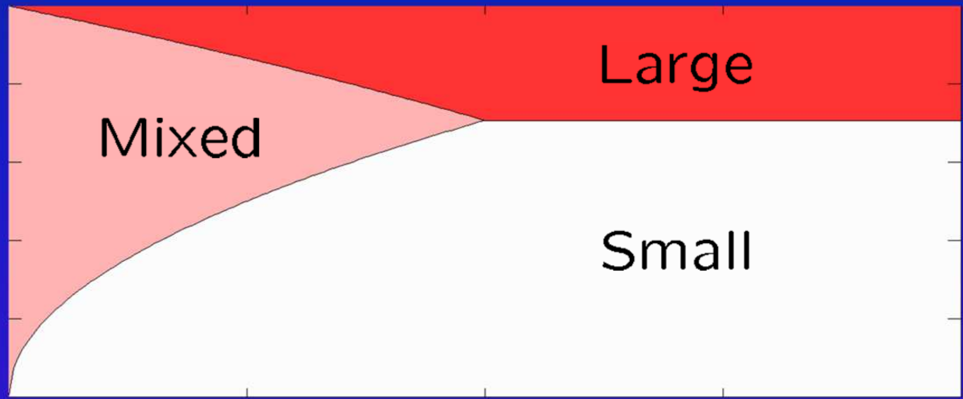
$$\psi = \int_0^z u(z')\,dz'$$

- this can be integrated to show there are three intersecting shocks for a homogeneous inflow with $\phi = \phi_0$

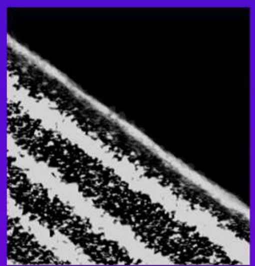
$$\psi_1 = S_{ls}\phi_0 x, \quad \psi_2 = 1 - S_{ls}(1-\phi_0)x, \quad \psi_3 = \phi_0$$

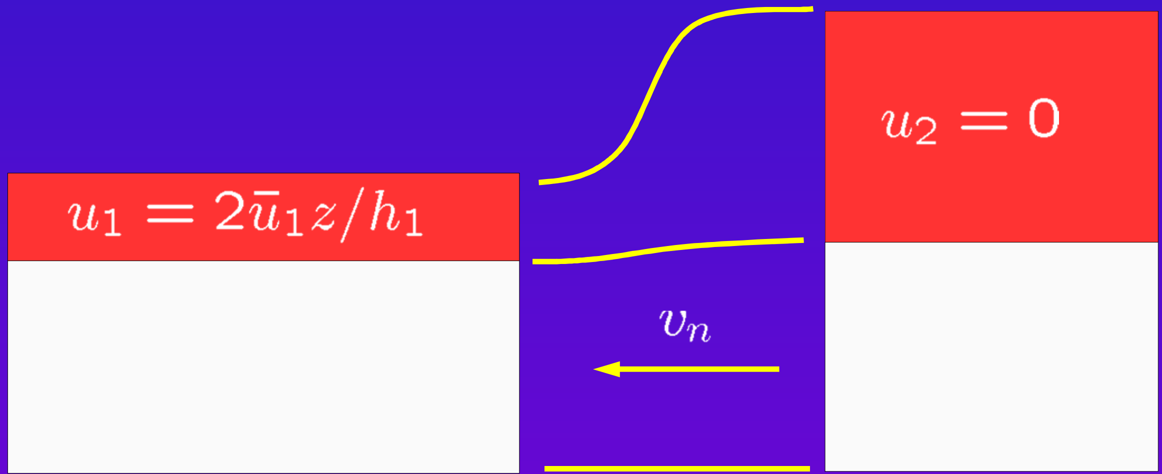
- within the flowing avalanche the particles “inverse-grade”

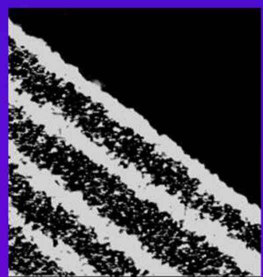





- when they stop the upper layers expand more
- because there is a higher mass flux at the top





A ternary mixture of large medium and small particles

- Theory gives three equations

$$\begin{aligned}
\frac{\partial \phi^l}{\partial t} + \nabla\cdot(\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{lm}\phi^l\phi^m + S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right)\\
\frac{\partial \phi^m}{\partial t} + \nabla\cdot(\phi^m \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{lm}\phi^m\phi^l + S_{ms}\phi^m\phi^s) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^m}{\partial z}\right)\\
\frac{\partial \phi^s}{\partial t} + \nabla\cdot(\phi^s \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s\phi^m) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right)
\end{aligned}$$

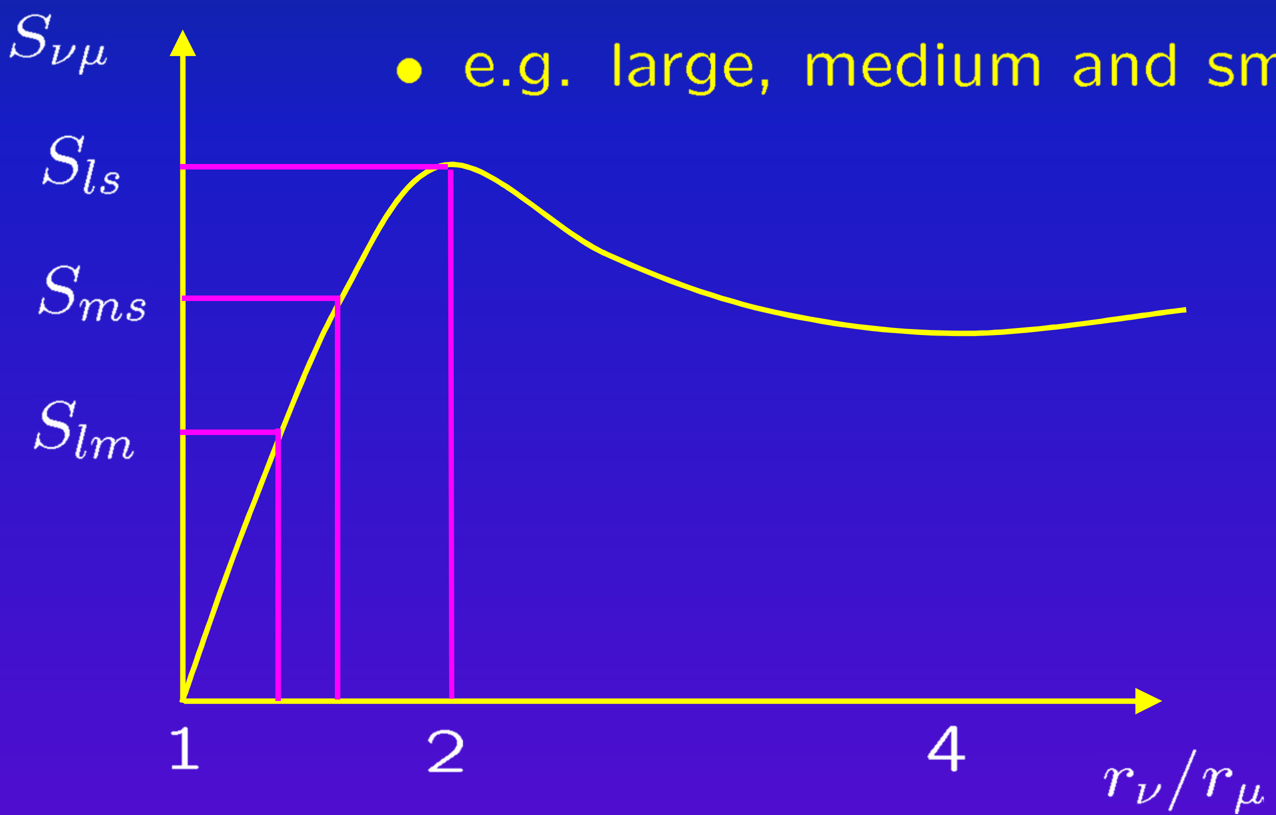
- Sum $\phi^l + \phi^m + \phi^s = 1$ allows $\phi^m$ to be eliminated

$$\begin{aligned}
\frac{\partial \phi^l}{\partial t} + \nabla\cdot(\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{lm}\phi^l(1-\phi^l-\phi^s) + S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right)\\
\frac{\partial \phi^s}{\partial t} + \nabla\cdot(\phi^s \boldsymbol{u}) + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1-\phi^l-\phi^s)) &= \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right)
\end{aligned}$$

- We will see diffusion is important to maintain well-posedness

## Use bi-disperse experiments to determine segregation rates

- e.g. large, medium and small particles




- Golick & Daniels (2009) suggest segregation rate may have maximum at a grain size ratio of two
- $\Rightarrow$ segregation rates $S_{ls}, S_{lm}, S_{ms}$ are not ordered

Use bi-disperse experiments to determine segregation rates

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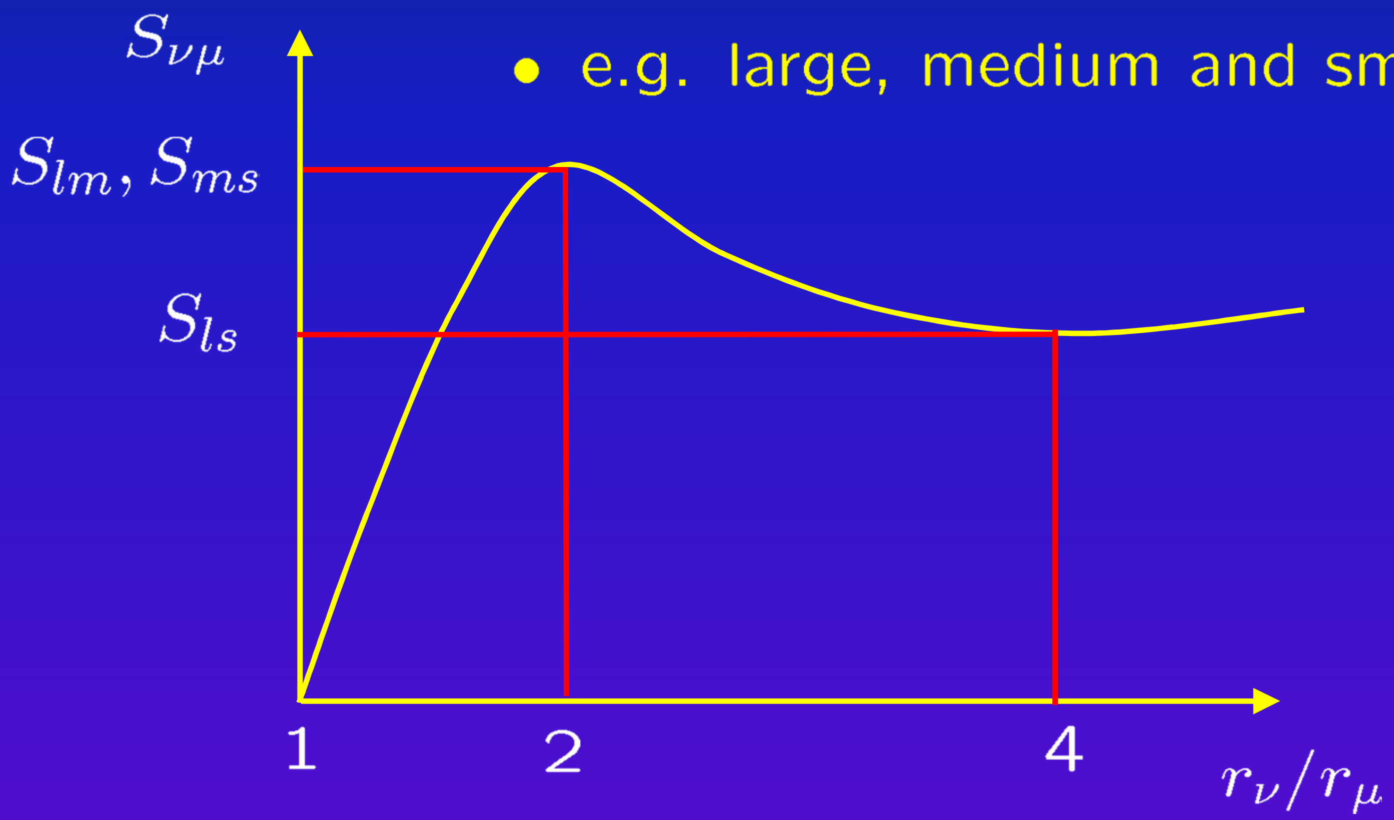



- Golick & Daniels (2009) suggest segregation rate may have maximum at a grain size ratio of two
- $\Rightarrow$ segregation rates $S_{ls}, S_{lm}, S_{ms}$ are not ordered

## Characteristic wave-speeds

- Defining $\phi = (\phi^l, \phi^m, \phi^s)^T$ system can be written as

$$\frac{D\boldsymbol{\phi}}{Dt} + \boldsymbol{A}\frac{\partial \boldsymbol{\phi}}{\partial z} = \boldsymbol{0},$$

$$\boldsymbol{A} = \begin{pmatrix} S_{lm}\phi^m + S_{ls}\phi^s & S_{lm}\phi^l & S_{ls}\phi^l \\ -S_{lm}\phi^m & -S_{lm}\phi^l + S_{ms}\phi^s & S_{ms}\phi^m \\ -S_{ls}\phi^s & -S_{ms}\phi^s & -S_{ls}\phi^l - S_{ms}\phi^m \end{pmatrix}.$$

- The characteristic wave speeds are given by

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0,$$

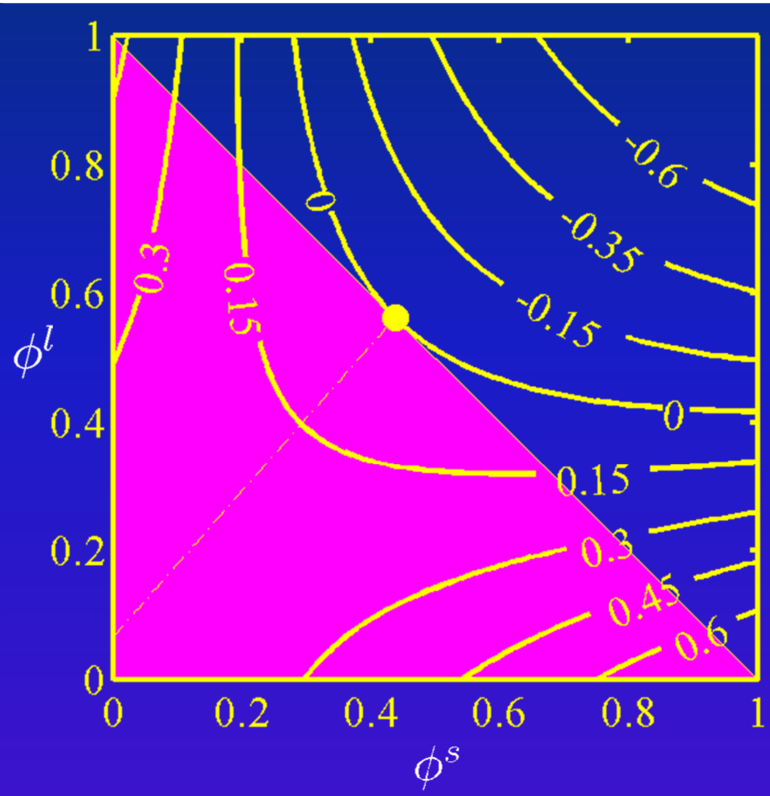
- The eigenvalue $\lambda = 0$ is easily spotted, which leaves

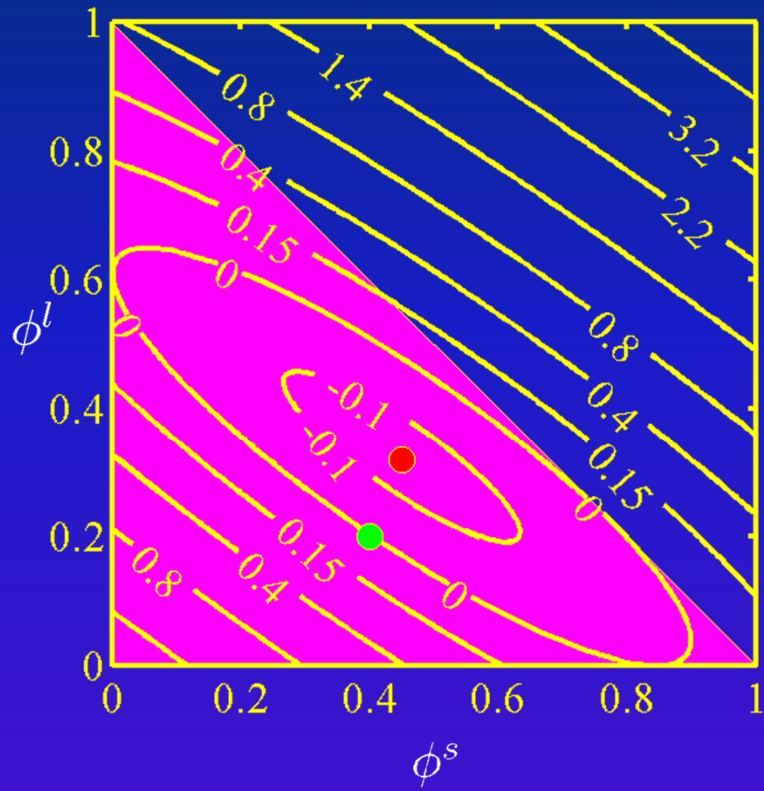
$$\lambda^2 + \gamma_1\lambda + \gamma_2 = 0,$$

- The two characteristic wave speeds are therefore

$$\lambda_{1,2} = (-\gamma_1 \pm \sqrt{\Delta})/2.$$

$\Omega_3$ $S_{ls} = 1,\ S_{lm} = 0.3,\ S_{ms} = 0.1$



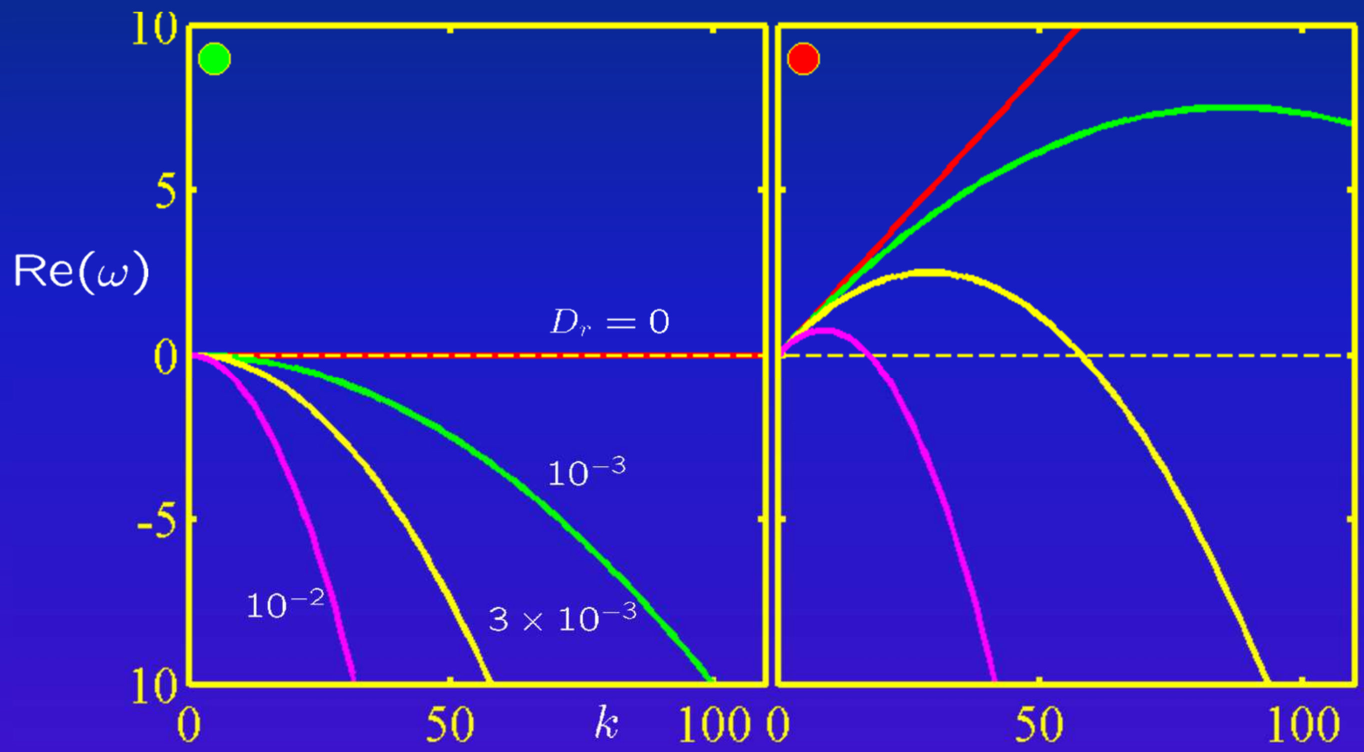

$\Omega_4$ $S_{ls} = 1/8,\ S_{lm} = 1,\ S_{ms} = 3/8$

- The discriminant

$$\Delta = \left[(S_{ls} - 2S_{lm} - S_{ms})\phi^l + (S_{ls} - S_{lm} - 2S_{ms})\phi^s + S_{lm} + S_{ms}\right]^2 - 4(S_{ls} - S_{ms})(S_{ls} - S_{lm})\phi^l\phi^s.$$

- When this is positive there are two real distinct roots and system is strictly hyperbolic

| | | | | |
|---|---|---|---|---|
| $\Omega_1:$ | $S_{ms} \geq S_{ls} \geq S_{lm},$ | | | hyperbolic |
| $\Omega_2:$ | $S_{lm} \geq S_{ls} \geq S_{ms},$ | | | hyperbolic |
| $\Omega_3:$ | $S_{ls} > S_{lm}$ | and | $S_{ls} > S_{ms},$ | hyperbolic |
| $\Omega_4:$ | $S_{ls} < S_{lm}$ | and | $S_{ls} < S_{ms},$ | looses hyperbolicity |




- Linear stability about a constant base state $\phi^\nu = \phi_0^\nu$

$$\phi^\nu = \phi_0^\nu + C_\nu \exp(ikz + \omega t)$$

- The largest real root

$$\mathrm{Re}(\omega) = \begin{cases} -D_r k^2, & \Delta_0 \geq 0, \\ -D_r k^2 + \dfrac{k}{2}\sqrt{-\Delta_0}, & \Delta_0 < 0. \end{cases}$$

- If $D_r = 0 \Rightarrow$ Hadamard unstable and ill-posed in $\Omega_4$

## Time dependent diffuse solutions

- Homogeneous initial state with a small perturbation

$$\begin{aligned}\phi^l(z,0) &= \phi^l_0 + 0.01\sin(2\pi n z),\\ \phi^s(z,0) &= \phi^s_0 - 0.01\sin(2\pi n z),\end{aligned}$$

- Solve using a standard Galerkin finite element method

$$\begin{aligned}\frac{\partial\phi^l}{\partial t} + \frac{\partial}{\partial z}(S_{lm}\phi^l(1-\phi^l-\phi^s) + S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial\phi^l}{\partial z}\right)\\ \frac{\partial\phi^s}{\partial t} + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1-\phi^l-\phi^s)) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial\phi^s}{\partial z}\right)\end{aligned}$$

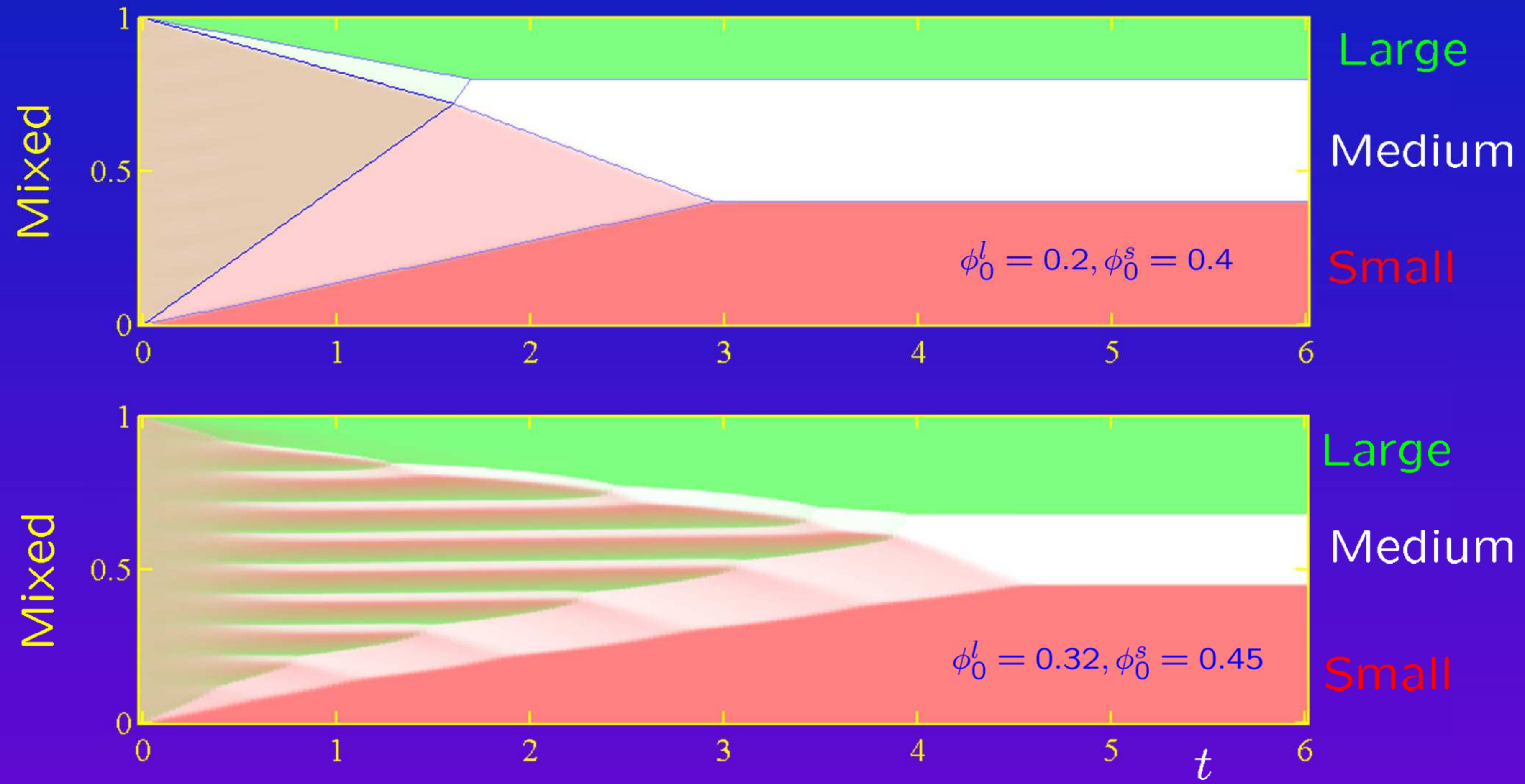
- subject to no-flux conditions

$$\mathcal{F}^\nu = -\sum_{\forall\mu} S_{\nu\mu}\phi^\nu\phi^\mu + D_r\frac{\partial\phi^\nu}{\partial z} = 0, \quad \text{on} \quad z = 0,1$$

- using Matlab inbuilt function pdepe

## Diffusive remixing regularizes the theory

$\Omega_4$: $S_{ls} = 1/8$, $S_{lm} = 1$, $S_{ms} = 3/8$, $D_r = 10^{-3}$




- Instabilities still develop when $\Delta_0 < 0$
- BUT are annihilated after a finite distance

## Two-dimensional steady-state solutions

- Homogeneous inflow at $x = 0$

$$\phi^\nu(0, z) = \phi_0^\nu$$

- with prescribed exponential downstream velocity field

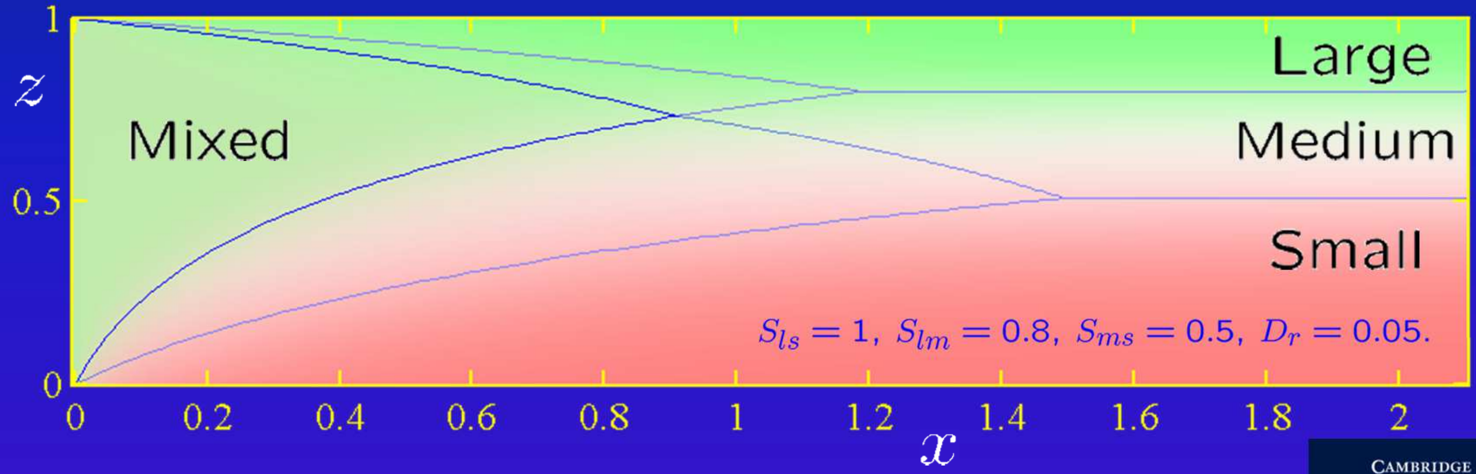
$$u(z) = \frac{\beta \exp(\beta z)}{\exp(\beta) - 1}, \qquad \beta > 0.$$

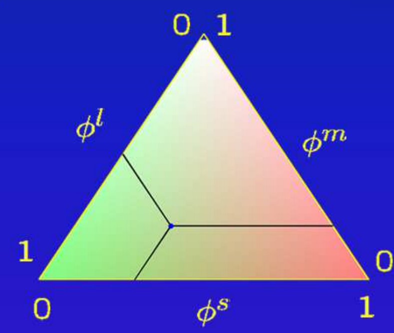
- Solve the two-dimensional steady problem

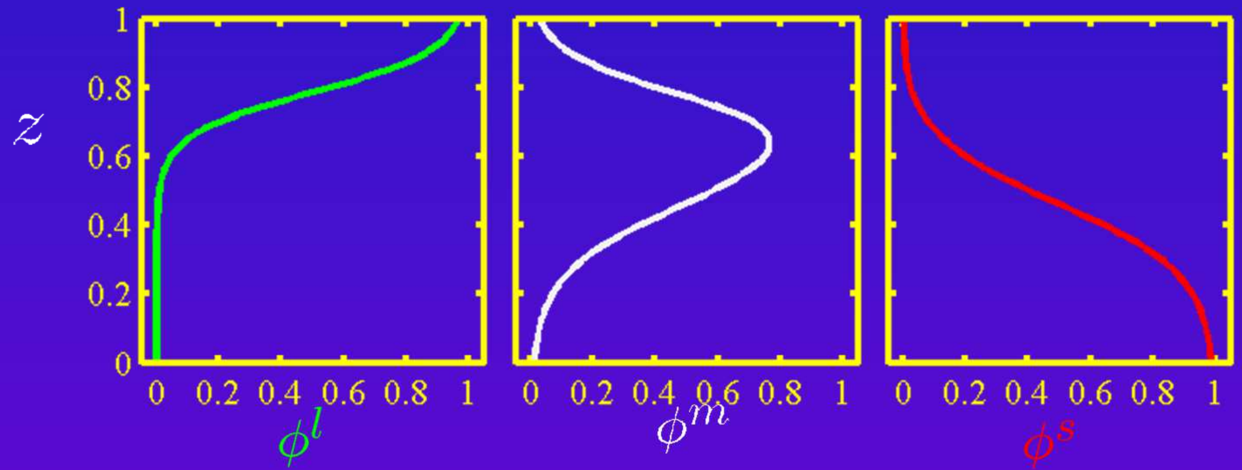
$$\begin{aligned} u\frac{\partial \phi^l}{\partial x} + \frac{\partial}{\partial z}(S_{lm}\phi^l(1-\phi^l-\phi^s) + S_{ls}\phi^l\phi^s) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^l}{\partial z}\right) \\ u\frac{\partial \phi^s}{\partial x} + \frac{\partial}{\partial z}(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1-\phi^l-\phi^s)) &= \frac{\partial}{\partial z}\left(D_r\frac{\partial \phi^s}{\partial z}\right) \end{aligned}$$

- subject to no-flux conditions at $z = 0, 1$
- using Matlab inbuilt function pdepe (Galerkin Method)

## Reverse distribution grading



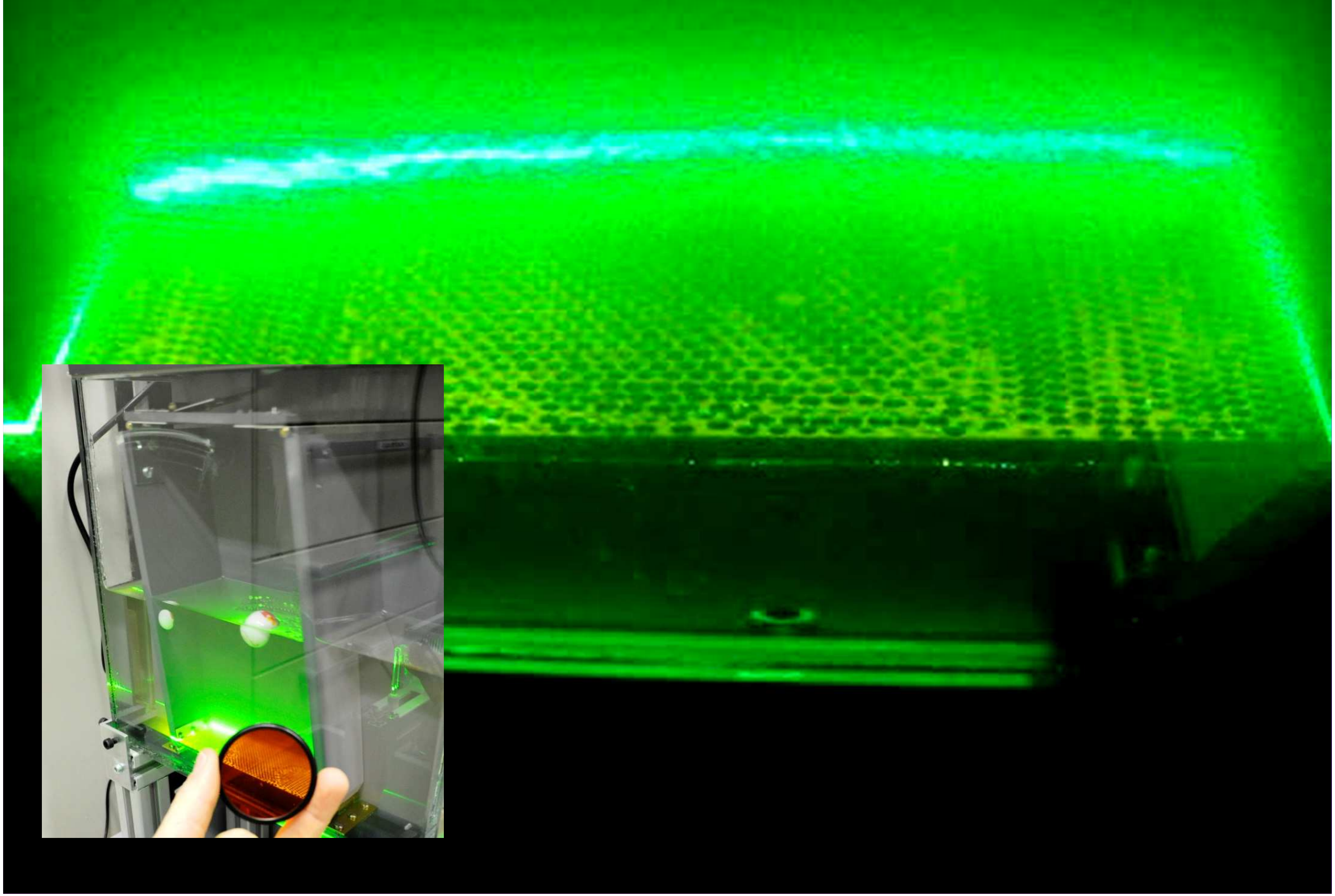


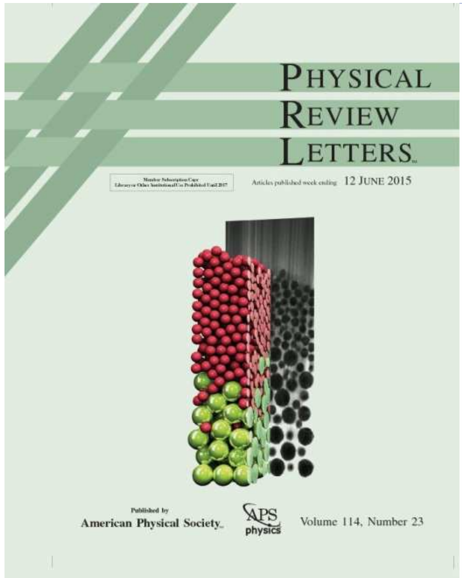
- comparable to experiments ...

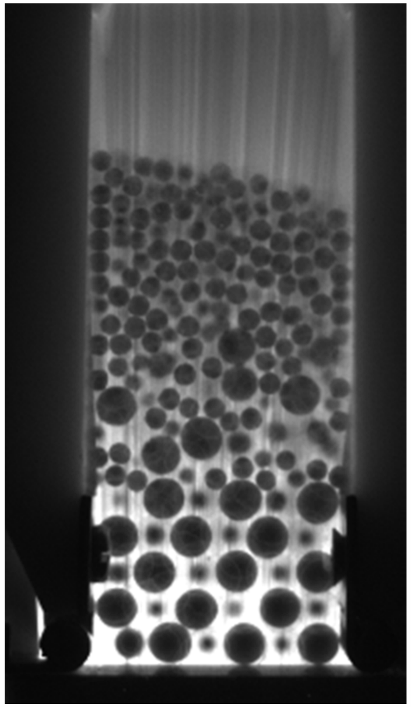
  Gray & Ancey (2011) *J. Fluid Mech.* **678**, 535-558
  Wiederseiner *et al.* (2011) *Phys. Fluids* **23**, 013301

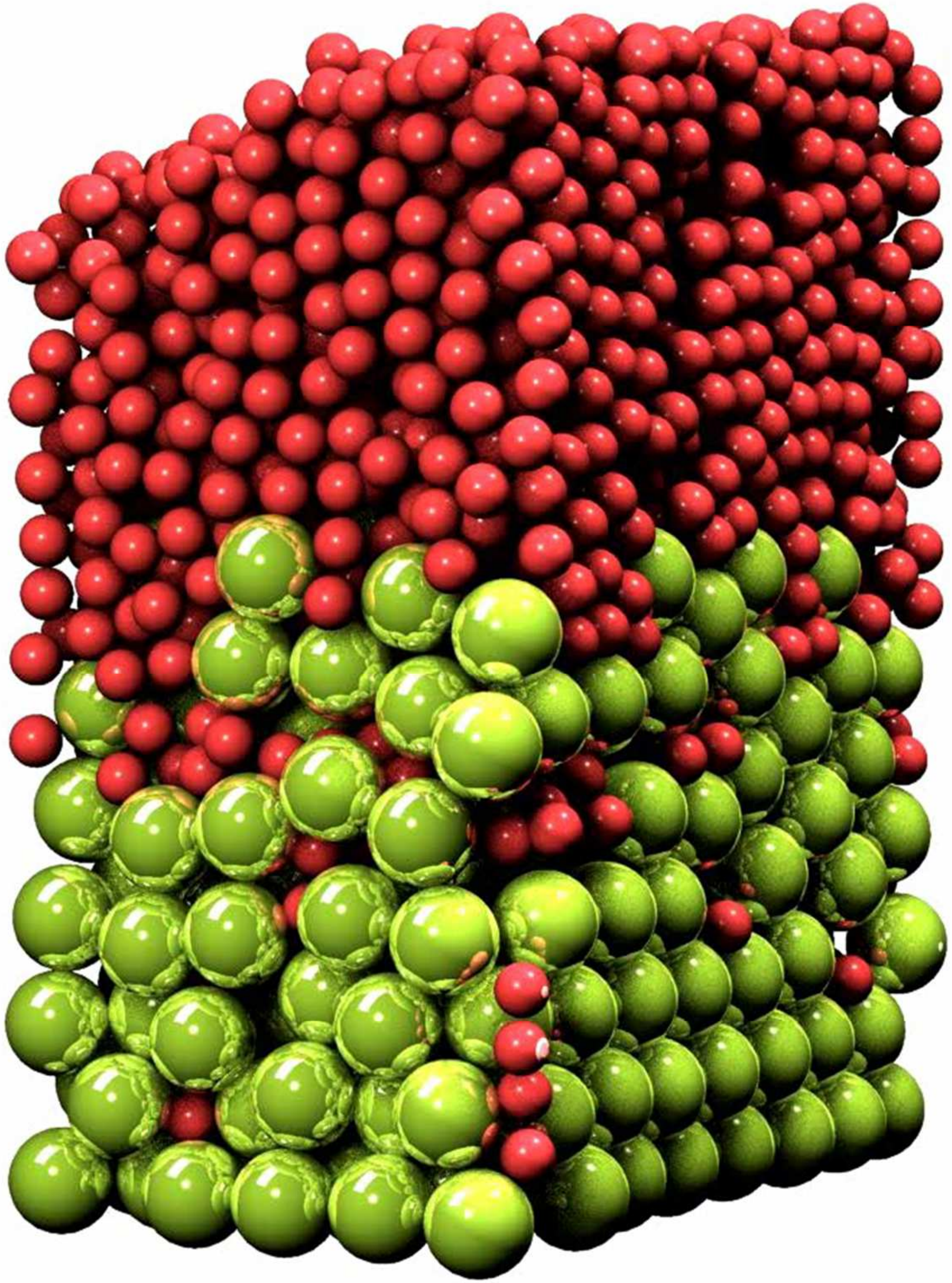
- and DEM simulations (Jim McElwaine ...)

We can look inside granular materials using index matching

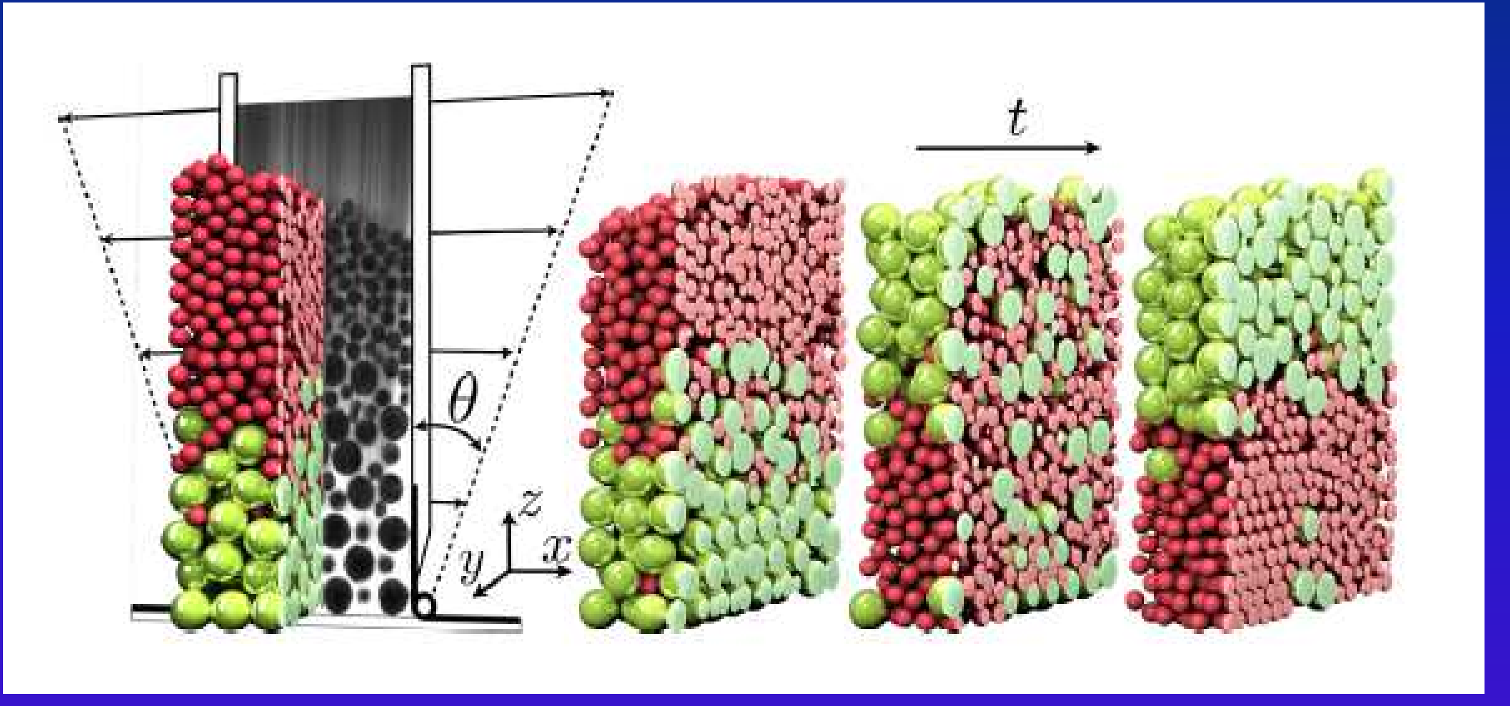




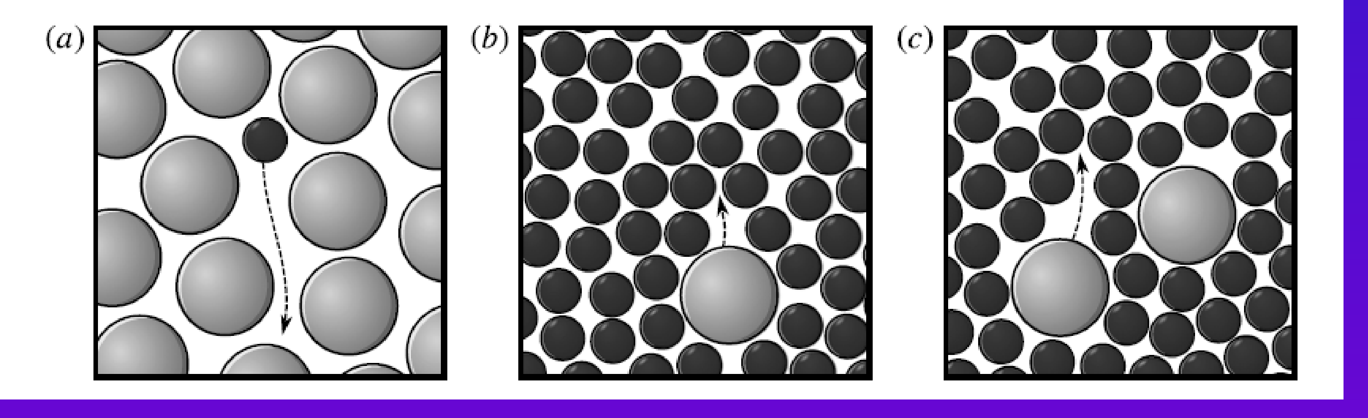





van der Vaart, Gajjar, Epely-Chauvin, Andreini, Gray & Ancey (2015) *Phys. Rev Lett.* **114**, 238001

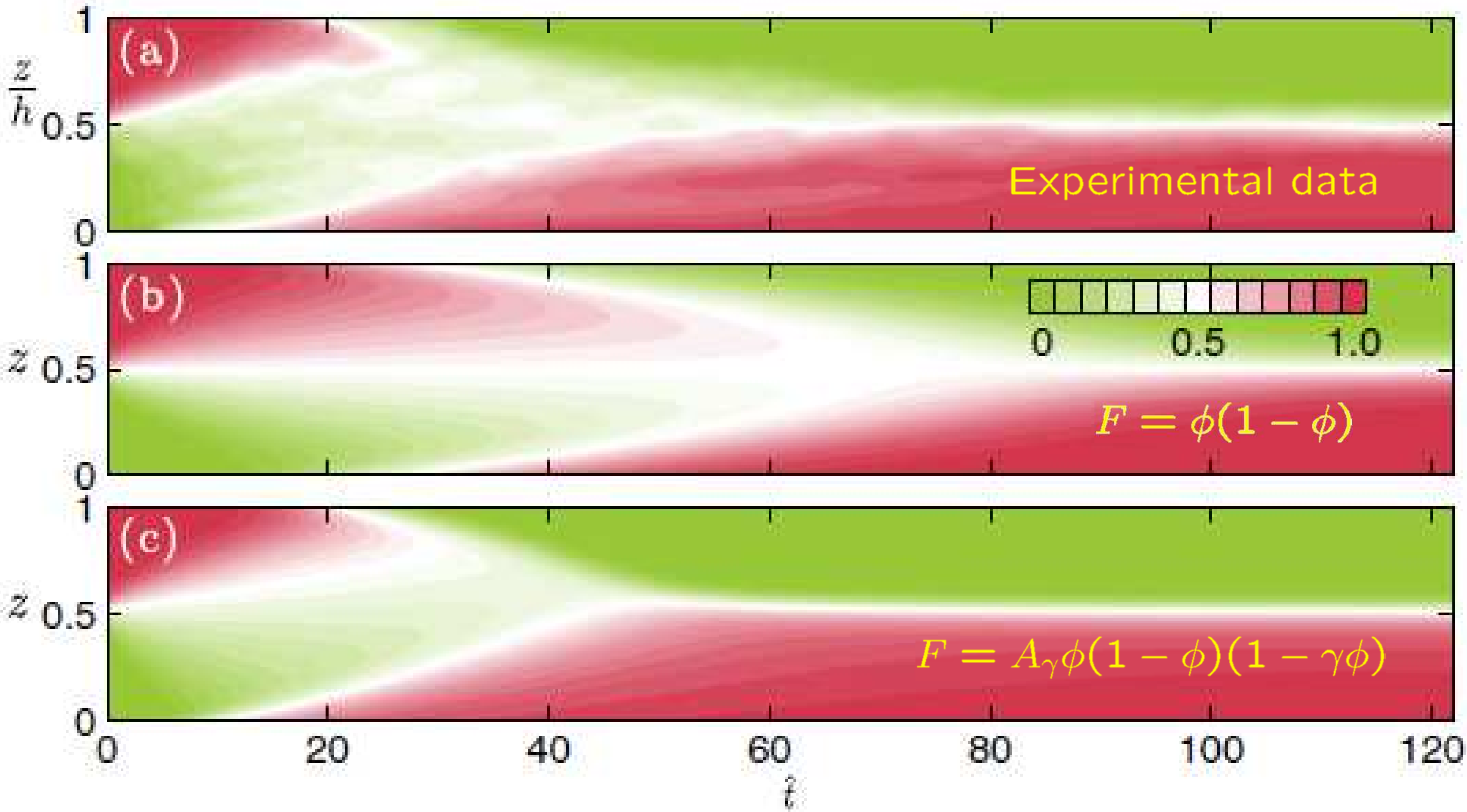



- a single large particle surrounded by fine rises slower than a single fine grain percolates through a matrix of large



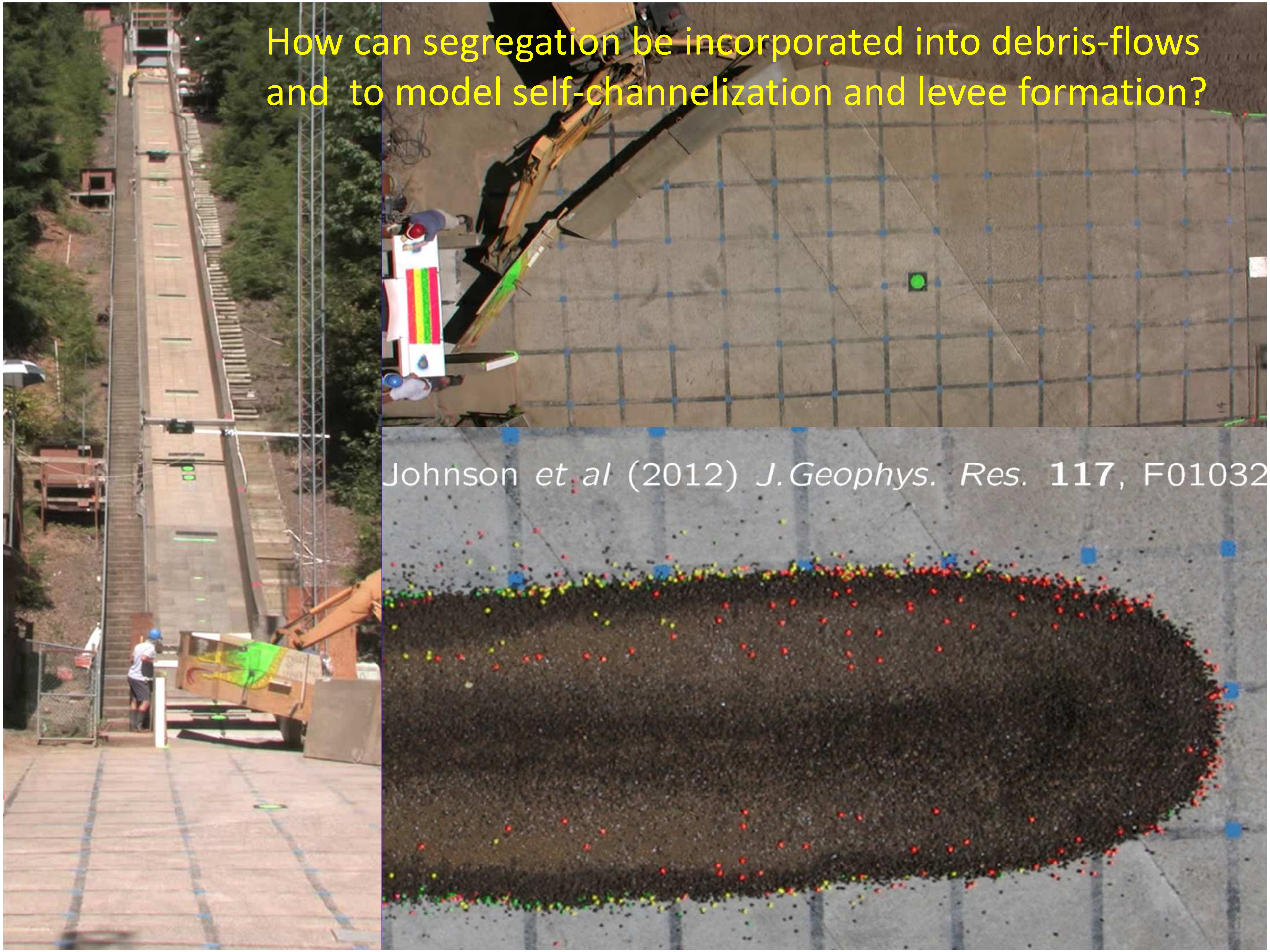

Gajjar & Gray (2014) J. Fluid Mech. 757 , 297-329.
van der Vaart *et al.* (2015) *Phys. Rev Lett.* **114**, 238001

This can be modelled using asymmetric segregation flux models

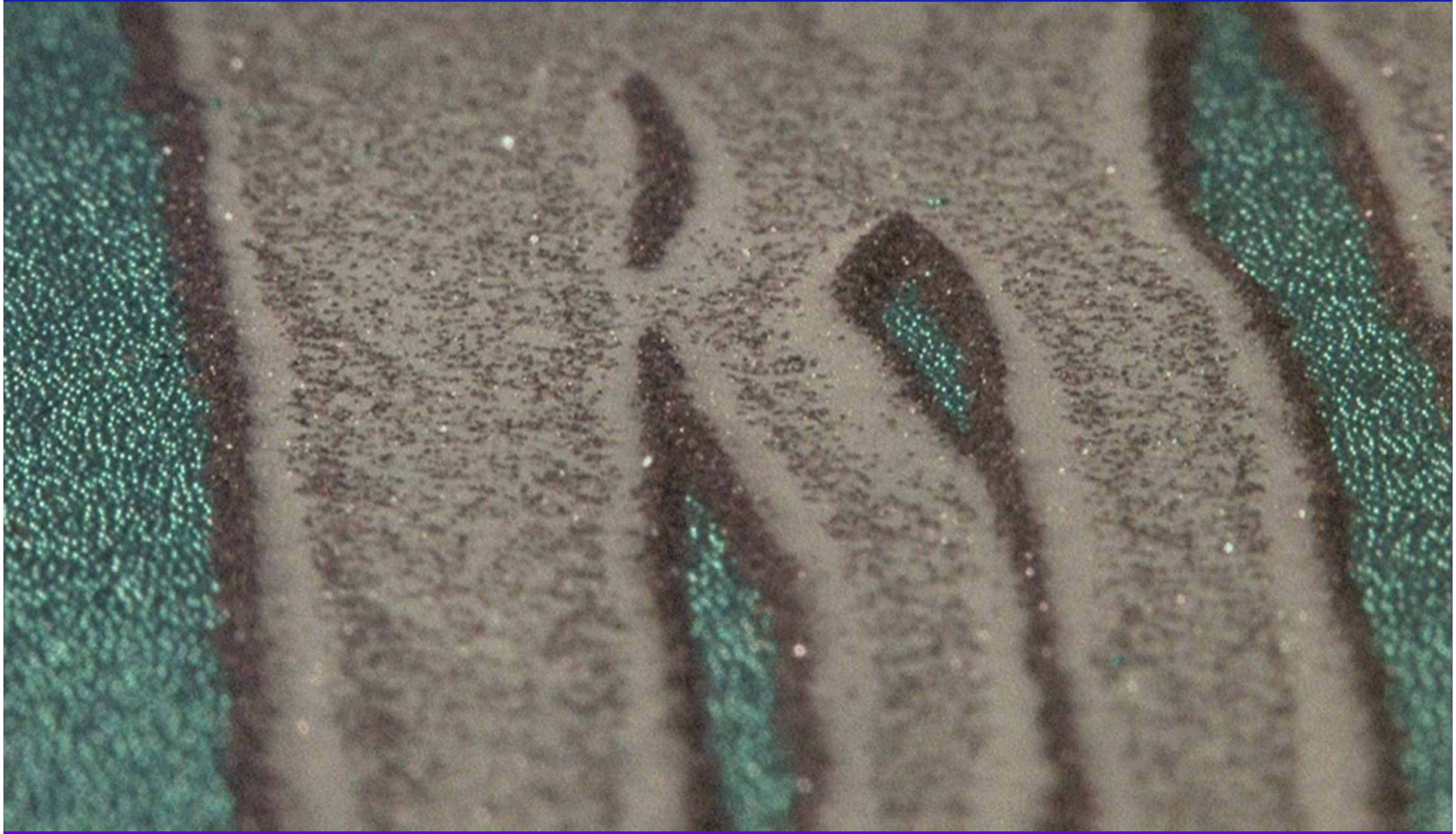



- which skew the maximum of the flux curves towards lower concentrations of fines

How can segregation be incorporated into debris-flows and to model self-channelization and levee formation?

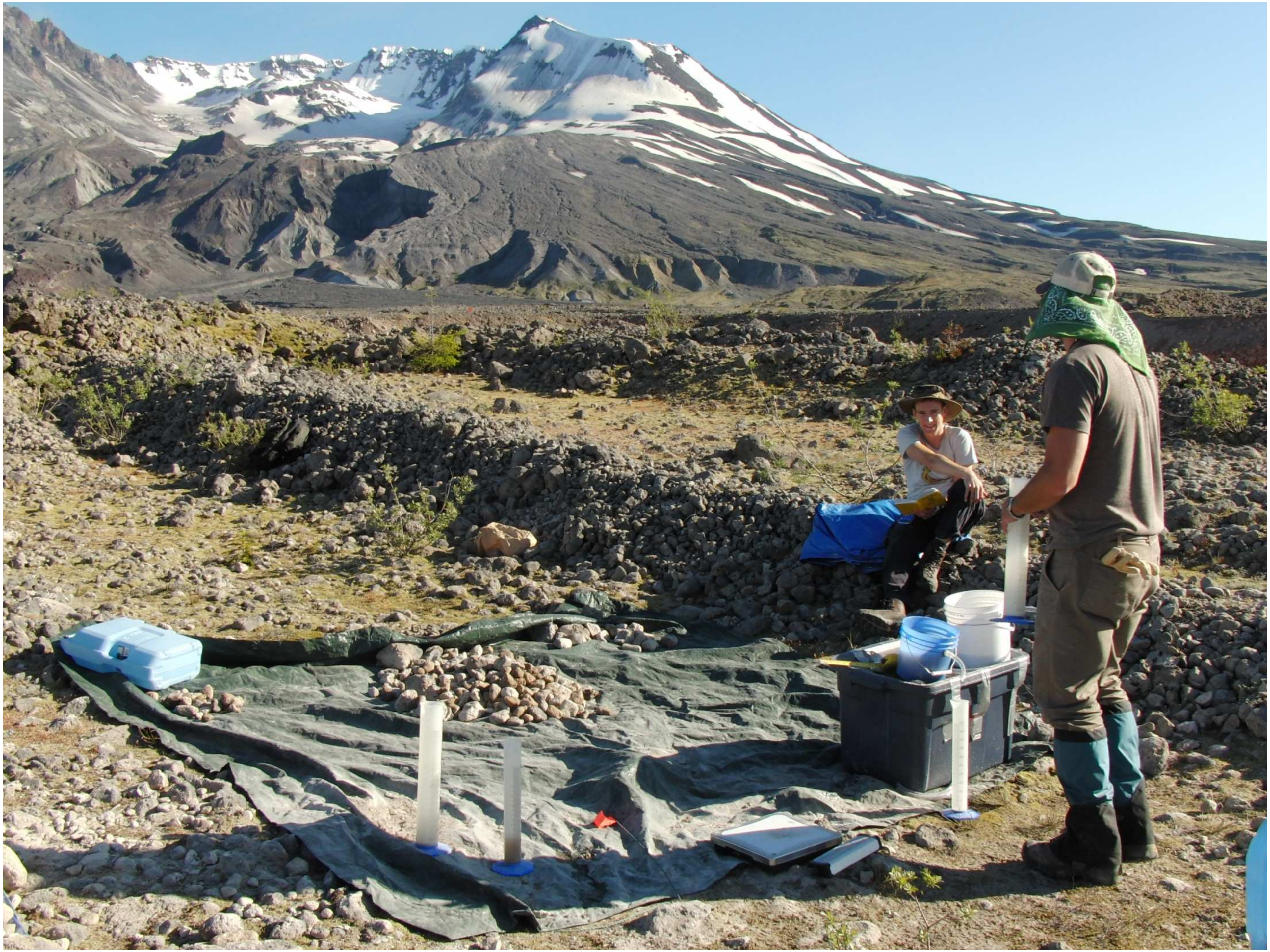


… as well as capture segregation-induced fingering instabilities ….



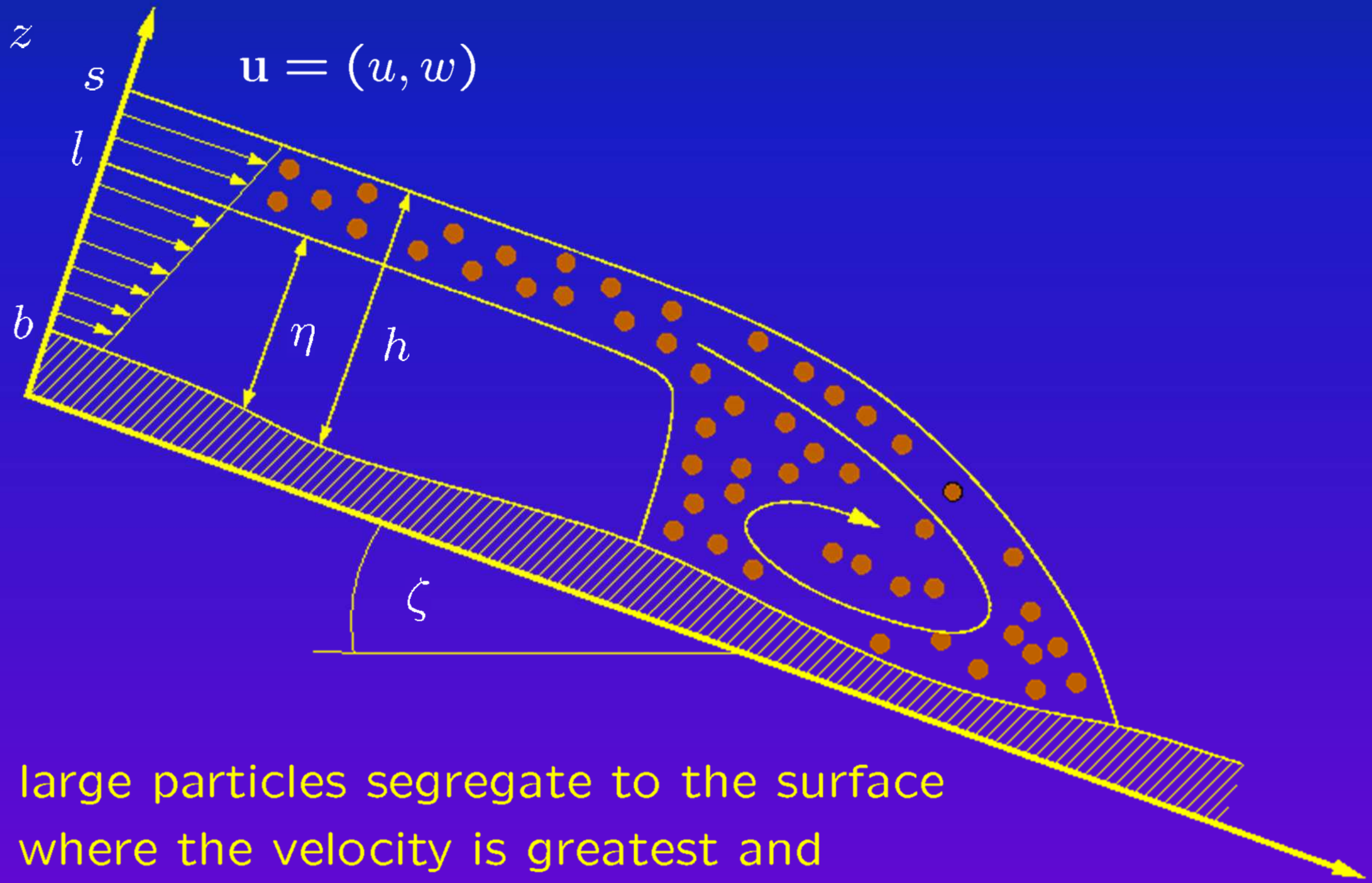
Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.


Mount St Helens, USA, 1980
Finer grained interior
Coarse rich levee
• segregation occurs in many hazardous natural flows
– debris-flows, pyroclastic flows & snow avalanches
• and leads to spontaneous flow organization and longer run-out
Mt St Helens July 22, 1980
USGS




Mt St Helens July 22, 1980

## Transport and accumulation of large particles



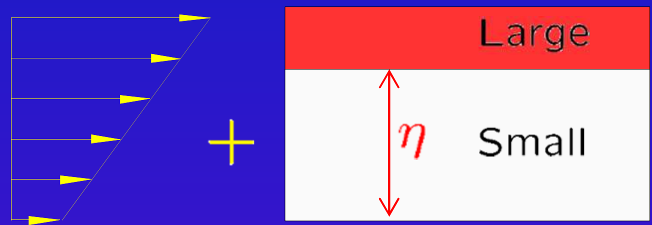

- large particles segregate to the surface
- where the velocity is greatest and
- are transported to the flow front where they are
- over run and recirculated by particle size segregation

## A depth averaged theory for particle size segregation

- Integrating the segregation-remixing equation w.r.t $z$
- subject to the no flux and kinematic boundary conditions gives

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\overline{\phi u}) = 0$$

- where the integrals evaluated assuming




i.e. linear velocity with basal slip and sharp segregation

$$h\bar{\phi} = \int_b^s \phi^s \, dz = \eta$$

$$h\overline{\phi u} = \int_b^s \phi^s u \, dz = \eta\bar{u} - (1-\alpha)\bar{u}\eta\left(1 - \frac{\eta}{h}\right)$$

- This yields the large particle transport equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta\bar{u}) - \frac{\partial}{\partial x}\left((1-\alpha)\bar{u}\eta\left(1-\frac{\eta}{h}\right)\right) = 0.$$

- for the evolution of the inversely graded shock interface $\eta$.

Gray & Kokelaar (2010) *J. Fluid Mech.* **652**, 105–137

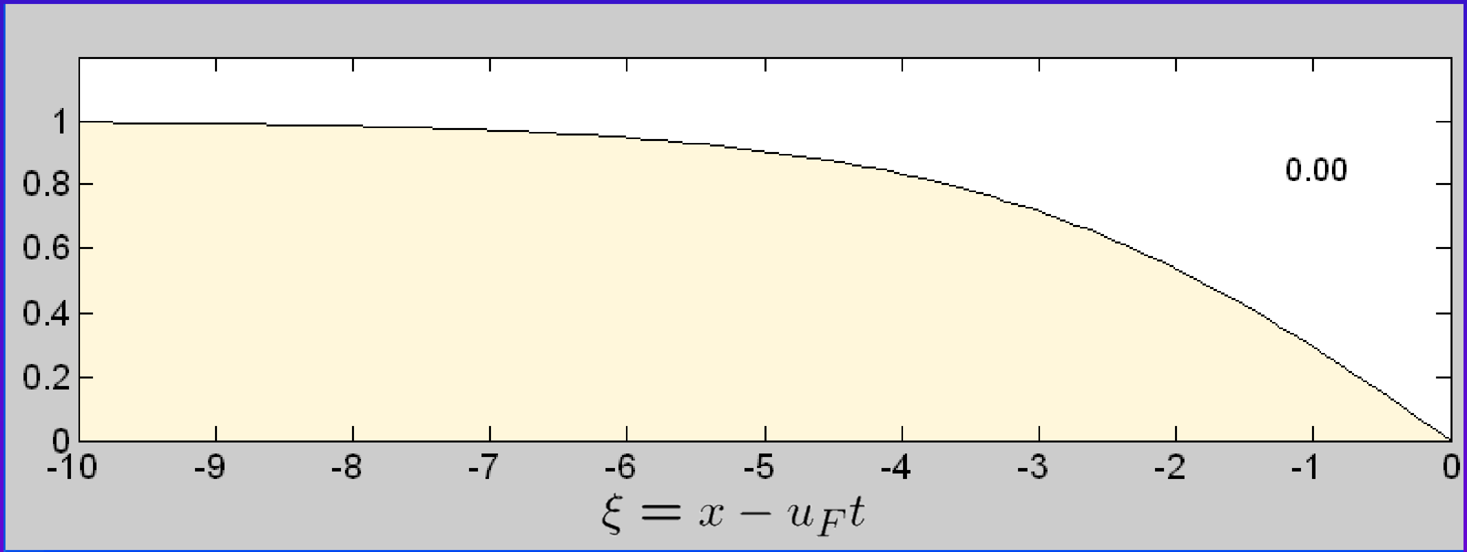
- Using $\eta = h\bar{\phi}$ this can also be rewritten as

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\bar{\phi}\bar{u}) - \frac{\partial}{\partial x}\left((1-\alpha)h\bar{u}\bar{\phi}\left(1-\bar{\phi}\right)\right) = 0.$$

- Remarkably similar to the segregation equation ...

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial z}(\phi w) - S_{ls}\frac{\partial}{\partial z}\left(\phi(1-\phi)\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi}{\partial z}\right)$$

- Large grains transported forwards to form bouldery flow front




- more RESISTIVE larger particles $\Rightarrow$ feedback on bulk flow

## A two-dimensional fully coupled model

- For avalanche thickness $h$, small particle thickness $\eta$ and depth-averaged velocity $\overline{\boldsymbol{u}}$ the 2D coupled model is

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0, \\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0, \\
\frac{\partial}{\partial t}(h\overline{u}) + \mathrm{div}(h\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}gh^2\cos\zeta\right) &= hg\boldsymbol{S},
\end{aligned}
$$

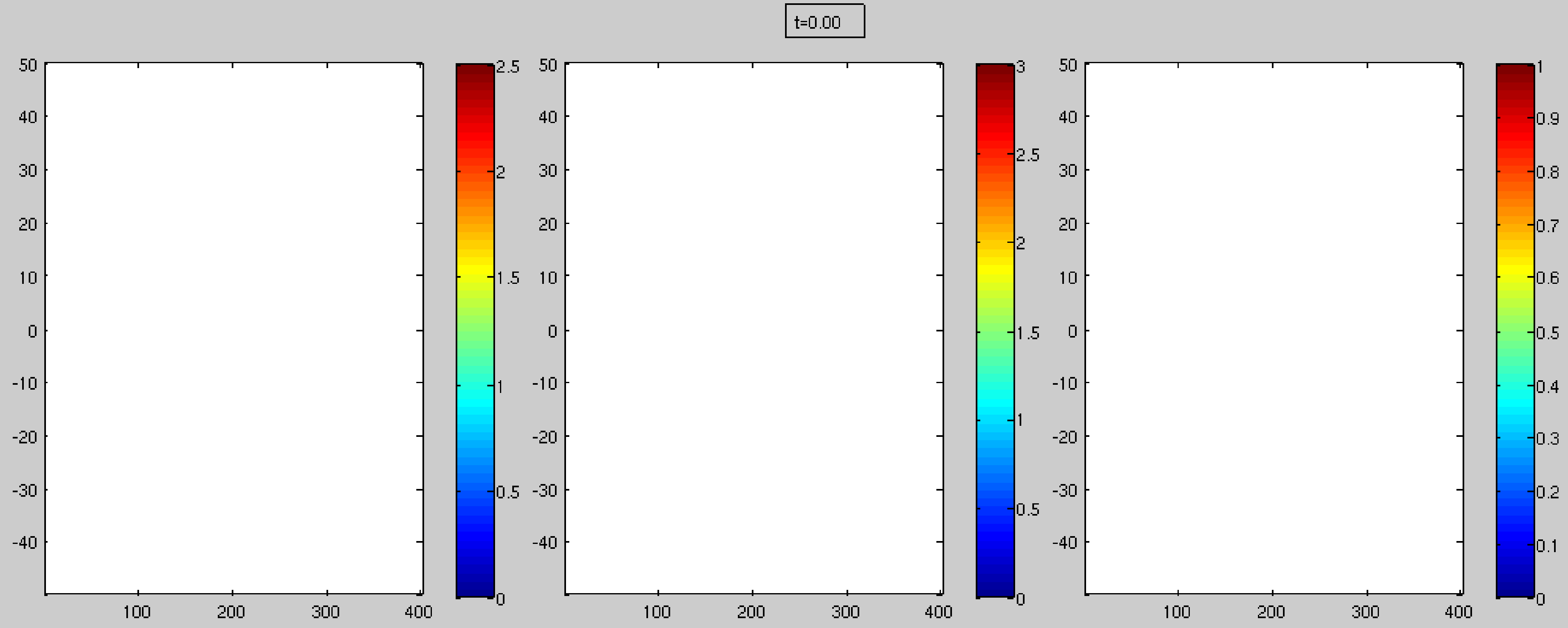
- source terms composed of gravity and basal friction

$$
\boldsymbol{S} = \begin{pmatrix} \sin\zeta - \mu(\overline{u}/|\overline{\boldsymbol{u}}|)\cos\zeta, \\ -\mu(\overline{v}/|\overline{\boldsymbol{u}}|)\cos\zeta, \end{pmatrix}
$$

- coupling through $\bar{\phi} = \eta/h$ dependent friction coefficient

$$
\mu = \left(1-\bar{\phi}\right)\mu^L + \bar{\phi}\mu^S
$$

## • Depth averaged coupled simulations ...

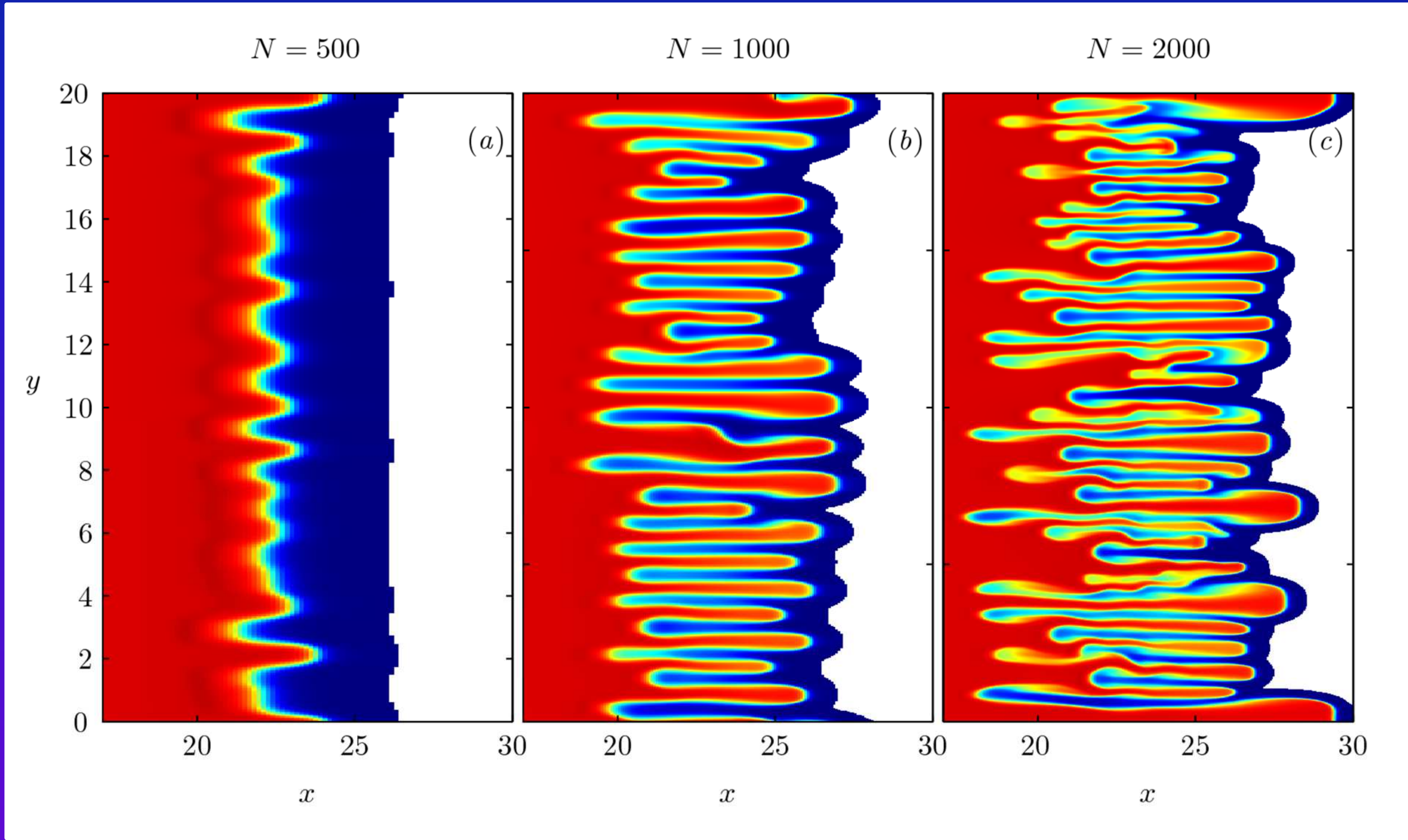



$h$ $|\overline{\boldsymbol{u}}|$ $\overline{\phi}$

- captures the instability mechanism, BUT model is too simple

Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.

Numerical solutions are grid dependent ...!




- Such ill-posed behaviour is an indication that some important physics is missing – in this case viscosity.

## A two-dimensional fully coupled model including rheology

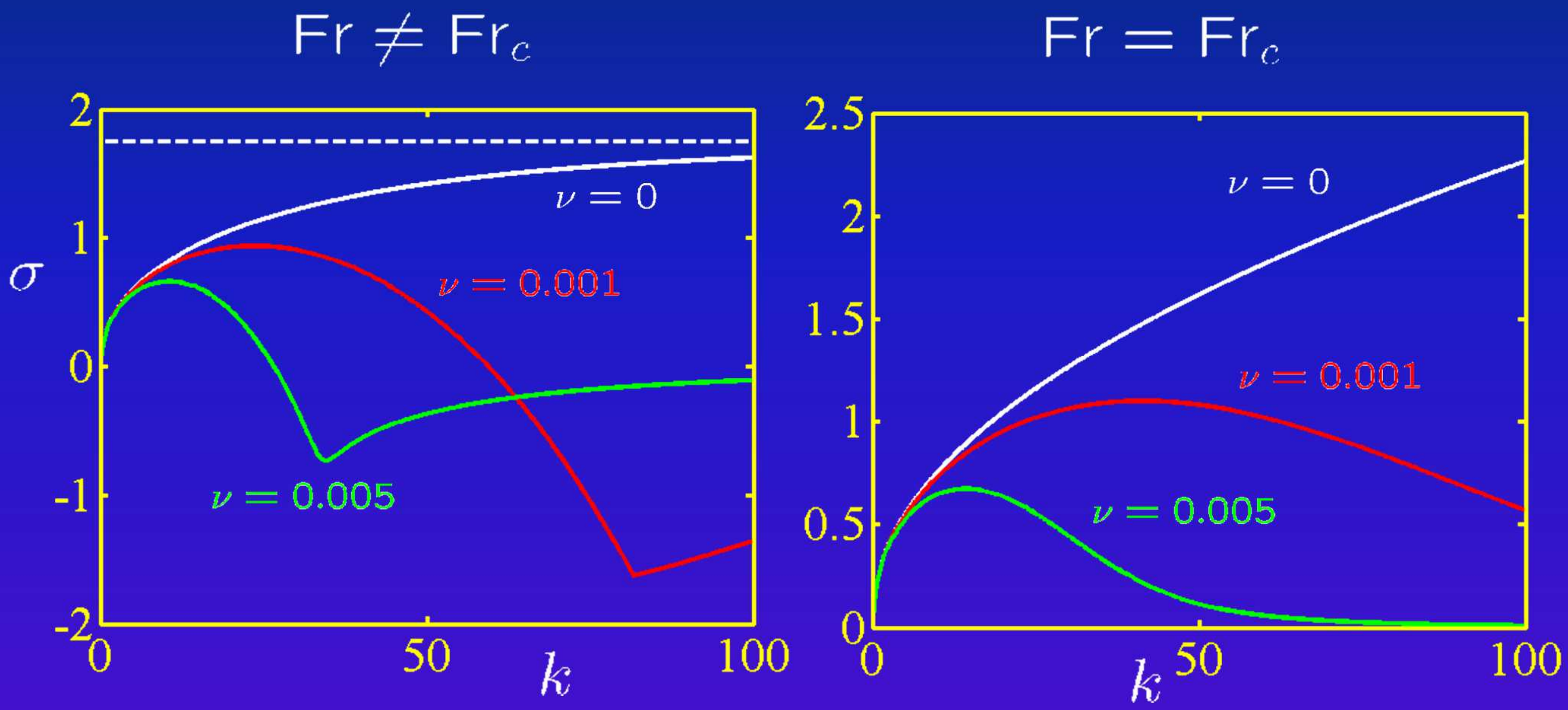
- When the depth-averaged $\mu(I)$-rheology is generalized to 2D it suggests a system of conservation laws

$$\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0, \\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0, \\
\frac{\partial}{\partial t}(h\bar{u}) + \mathrm{div}(h\overline{\boldsymbol{u}}\otimes\overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}gh^2\cos\zeta\right) &= hg\boldsymbol{S} + \mathrm{div}\left(\nu h^{\frac{3}{2}}\boldsymbol{D}\right),
\end{aligned}$$

- where the two-dimensional strain-rate tensor is

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right)$$

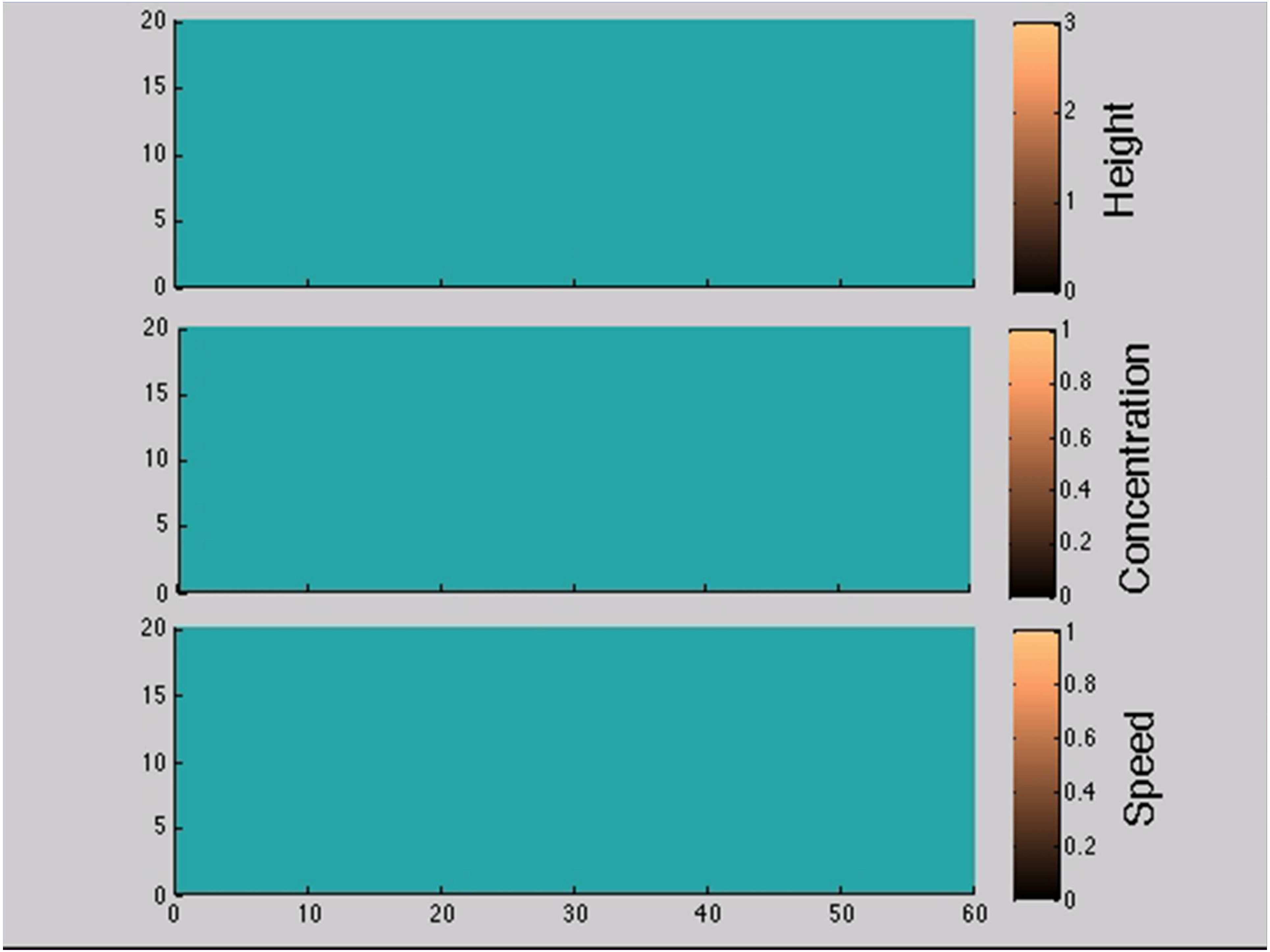
- and $\boldsymbol{L} = \mathrm{grad}(\bar{\boldsymbol{u}})$ is the depth-averaged velocity gradient
- Numerics converges … (Baker, Johnson & Gray in prep)




- characteristics coincide when

$$Fr = Fr_c = \frac{1}{(1-\alpha)|2\eta_0 - 1|}$$

- produces unbounded growth in inviscid case $\nu = 0$.
- The depth-averaged $\mu(I)$-rheology regularizes the equations


Height
Concentration
Speed
0
10
20
30
40
50
60
0
5
10
15
20
0
1
2
3
0.2
0.4
0.6
0.8