From particles to granular rheology

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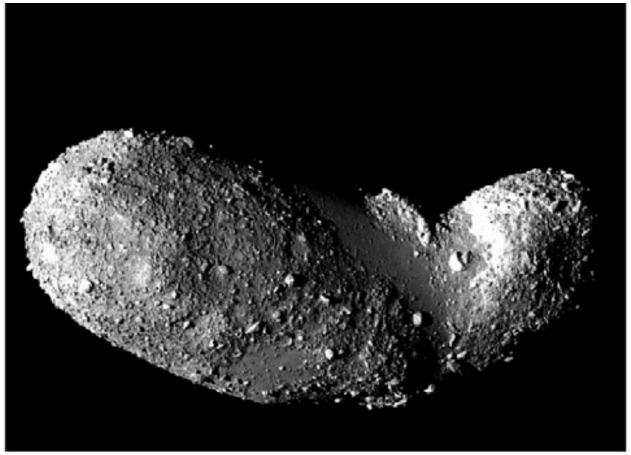
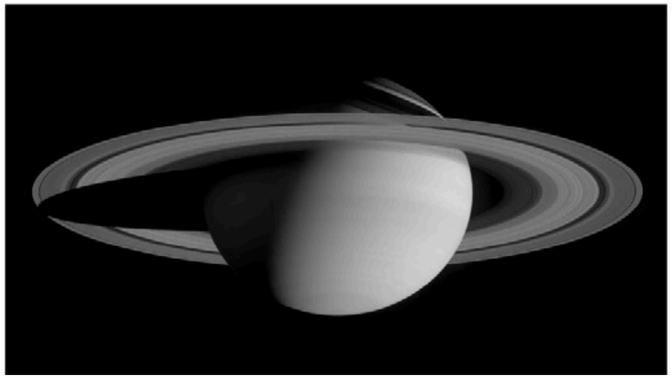
JMBC Particle Technology course – 4th March 2015

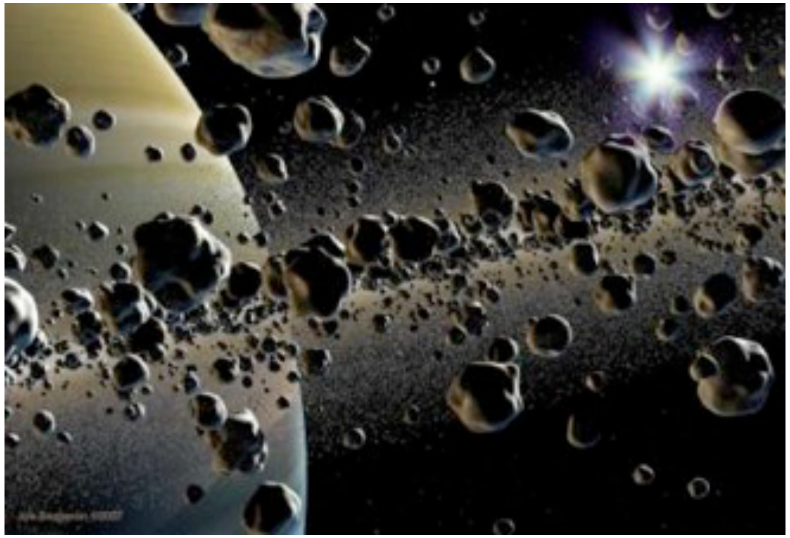


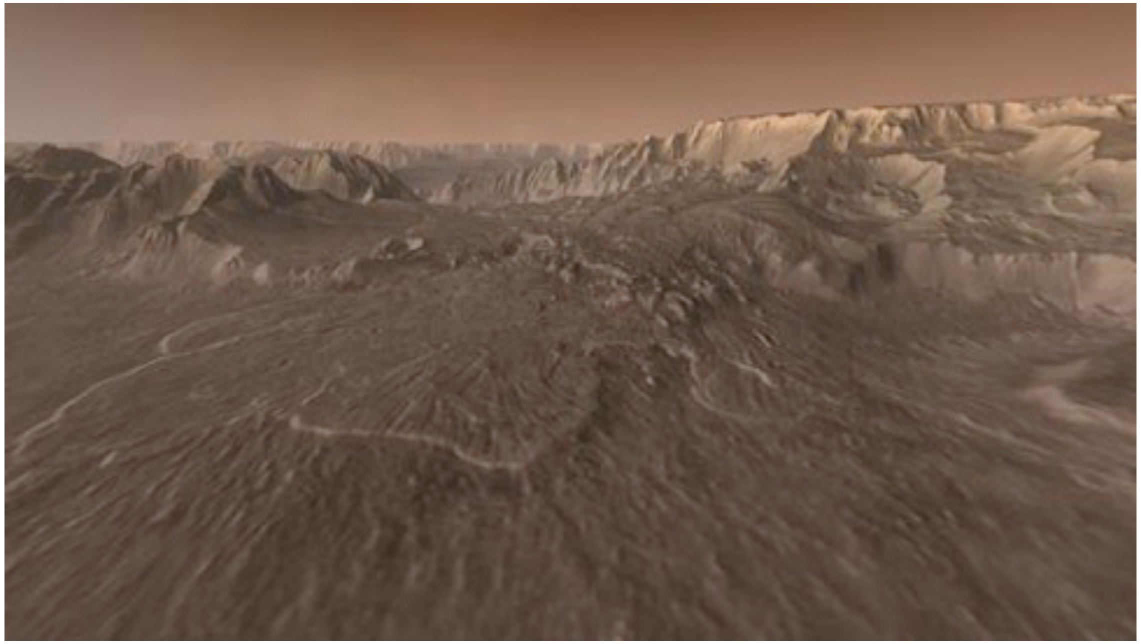




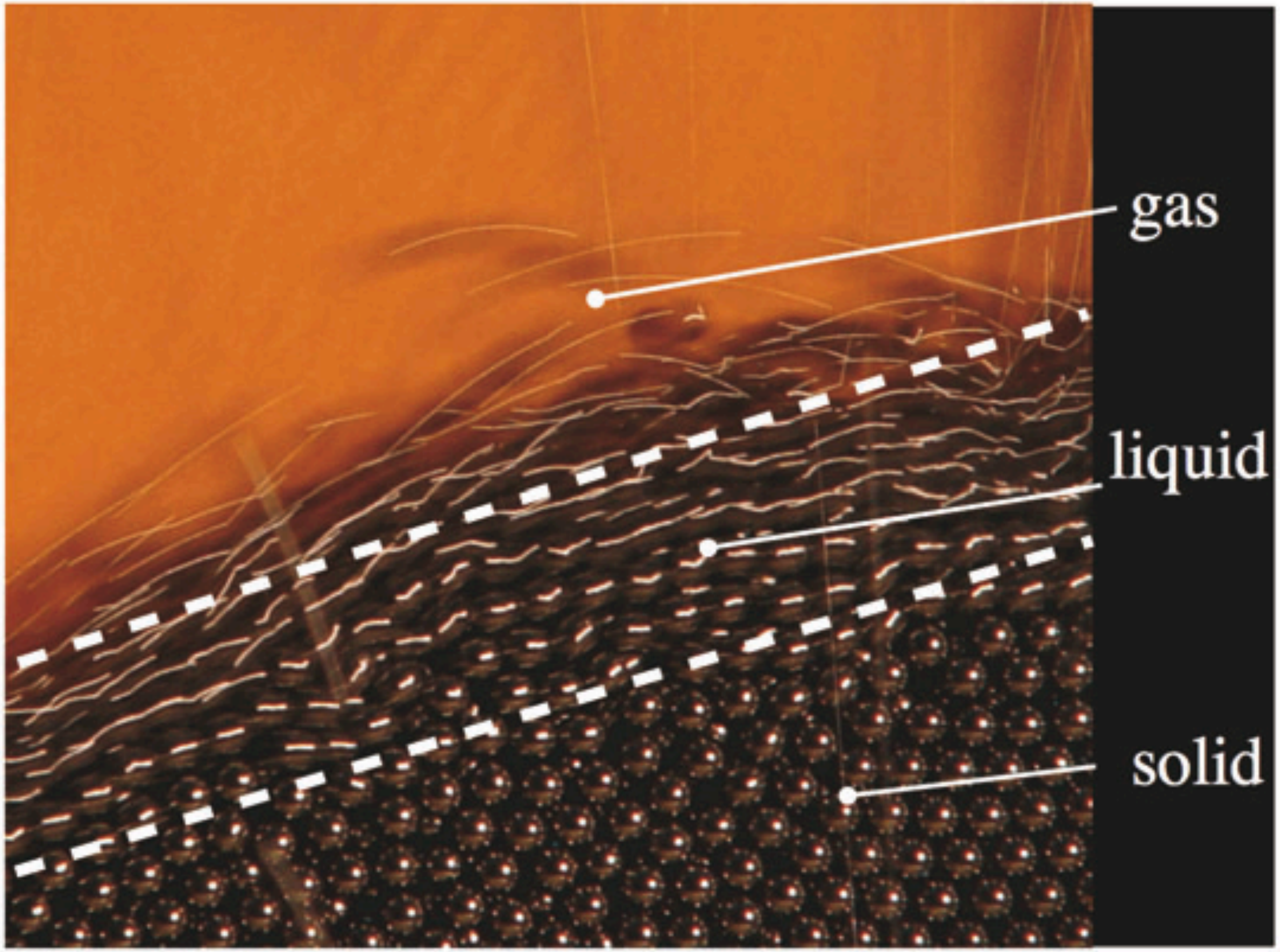




Granular material regimes



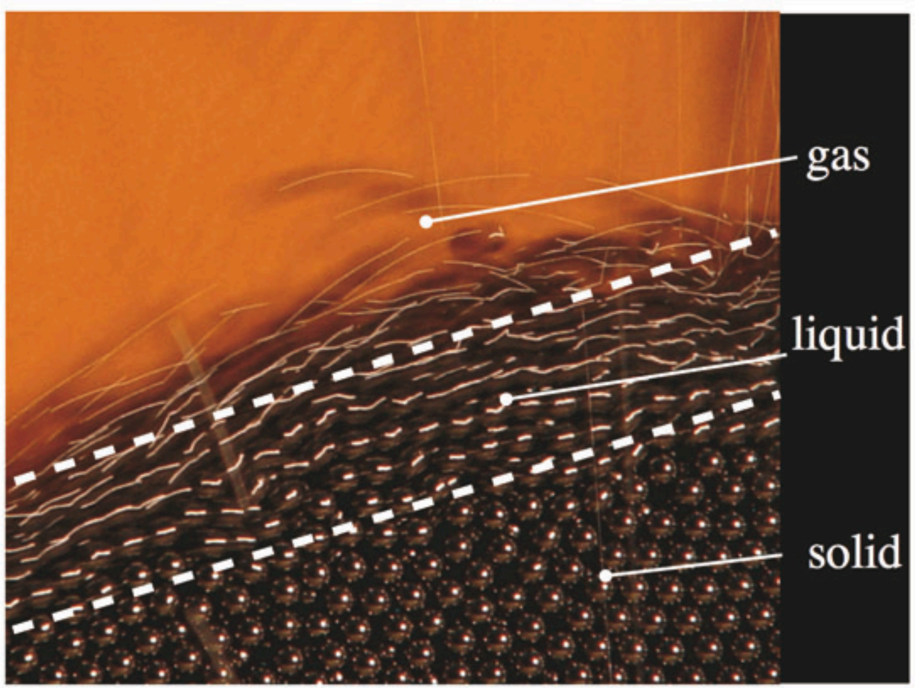


Solid+liquid+gas



[Forterre & Pouliquen, Annu. Rev. Fluid Mech. (2008)]

Solid+liquid+gas



Three regimes:

- **solid** – static
particles interact via frictional contacts

- **liquid** – dense, flow-like behavior
both collisions and friction

- **gas** – rapid dilute flow
particles interact via collisions

[Jaeger et al. (1996), Forterre & Pouliquen, Annu. Rev. Fluid Mech. (2008)]

Outline

- Introduction
- Internal force transmission
- Solid state
- Quasistatic regime and flow threshold
- Collisional and rapid granular flows
- Dense slow flows and inertial regime

Internal forces transmission

Contact forces

- Unique feature of granular material arises from internal force transmission

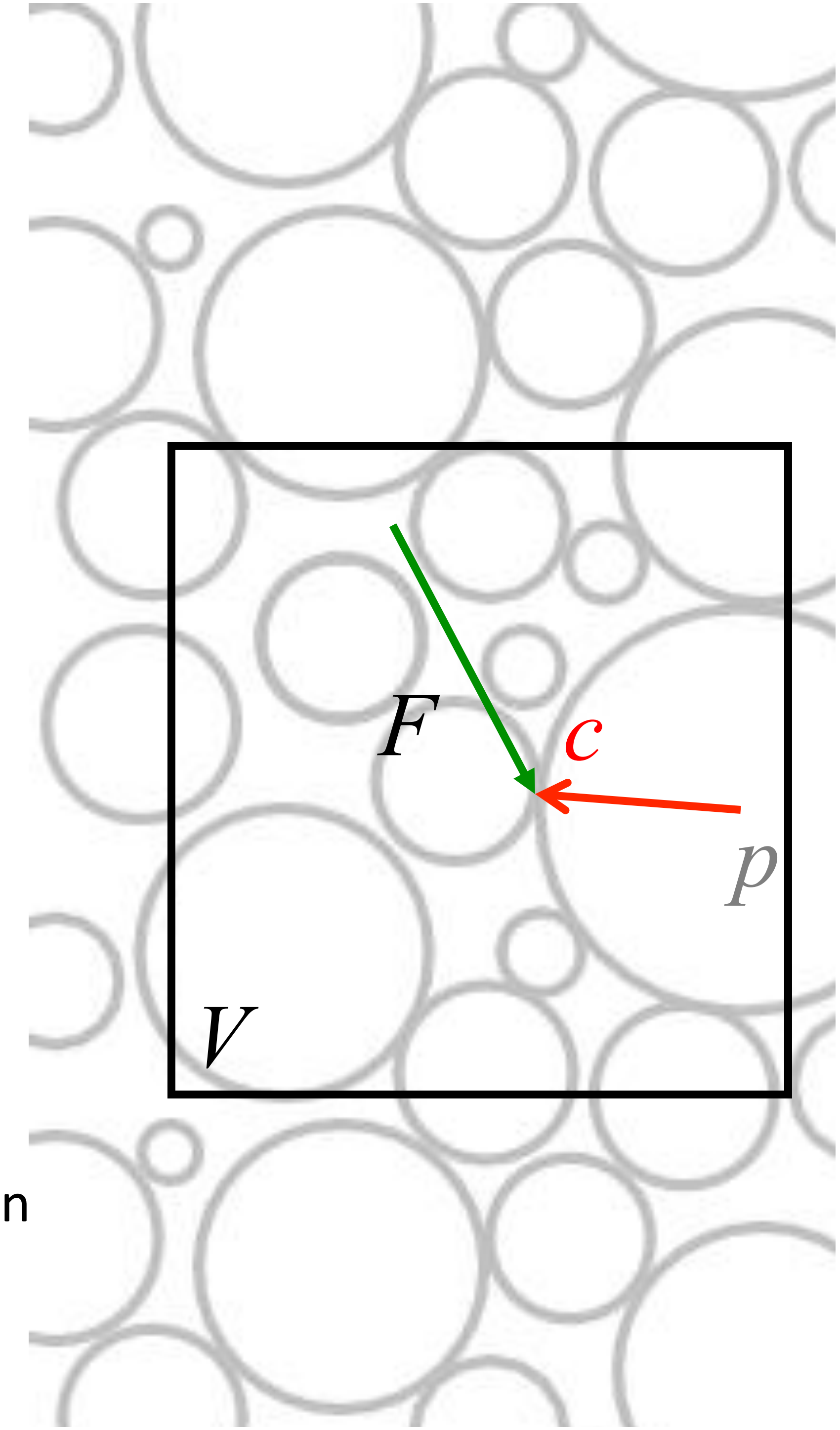
- Most fundamental microscopic property of granular materials: irreversible energy dissipation in the course of interaction collision between particles.

Micro-macro transition
Stress tensor

$$Q = \frac{1}{V}\sum_c \left(w_V^p\right) \boldsymbol{l}^{pc}\boldsymbol{F}^c$$

Any quantity:

- Scalar
- Vector
- Tensor: Stress



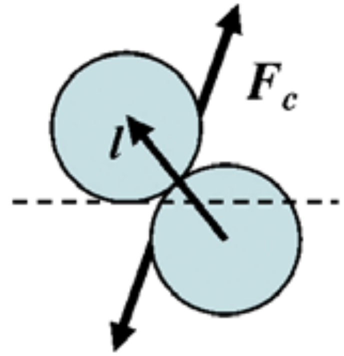
Overview of more complex formulations in [Weinhart et al. (2010)]

Stress tensor

a. Contact stress tensor

Due to the force transmission across interparticle forces

$$\sigma_{ij}^{c} = \frac{1}{V}\sum_{C=1}^{Nc} F_i^C l_j$$

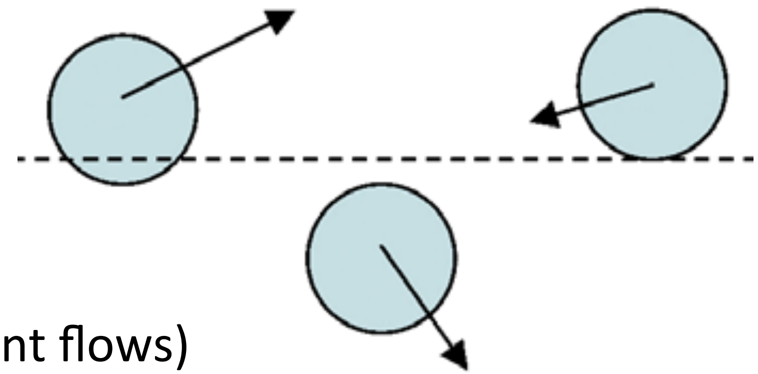


(a)

b. Streaming stress tensor

Due to the motion of a particle relative to the bulk material (Reynolds stress tensor in turbolent flows)

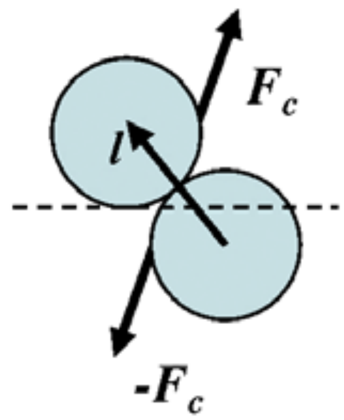
$$\sigma_{ij}^{s} = \frac{\rho_p \phi}{V}\sum_{p=1}^{Np} u'_i u'_j$$


(b)

Stress tensor

a. Contact stress tensor

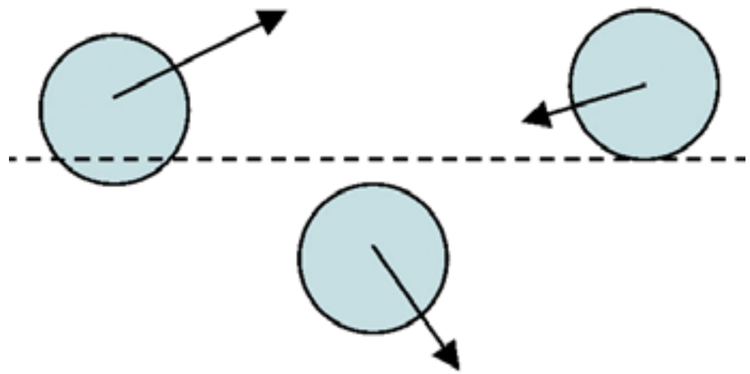
In hoppers, chutes, landslides: ϕ **> 50%**
is usually dominant in common granular flows



(a)

b. Streaming stress tensor

can usually be neglected


(b)

Hertzian contact law

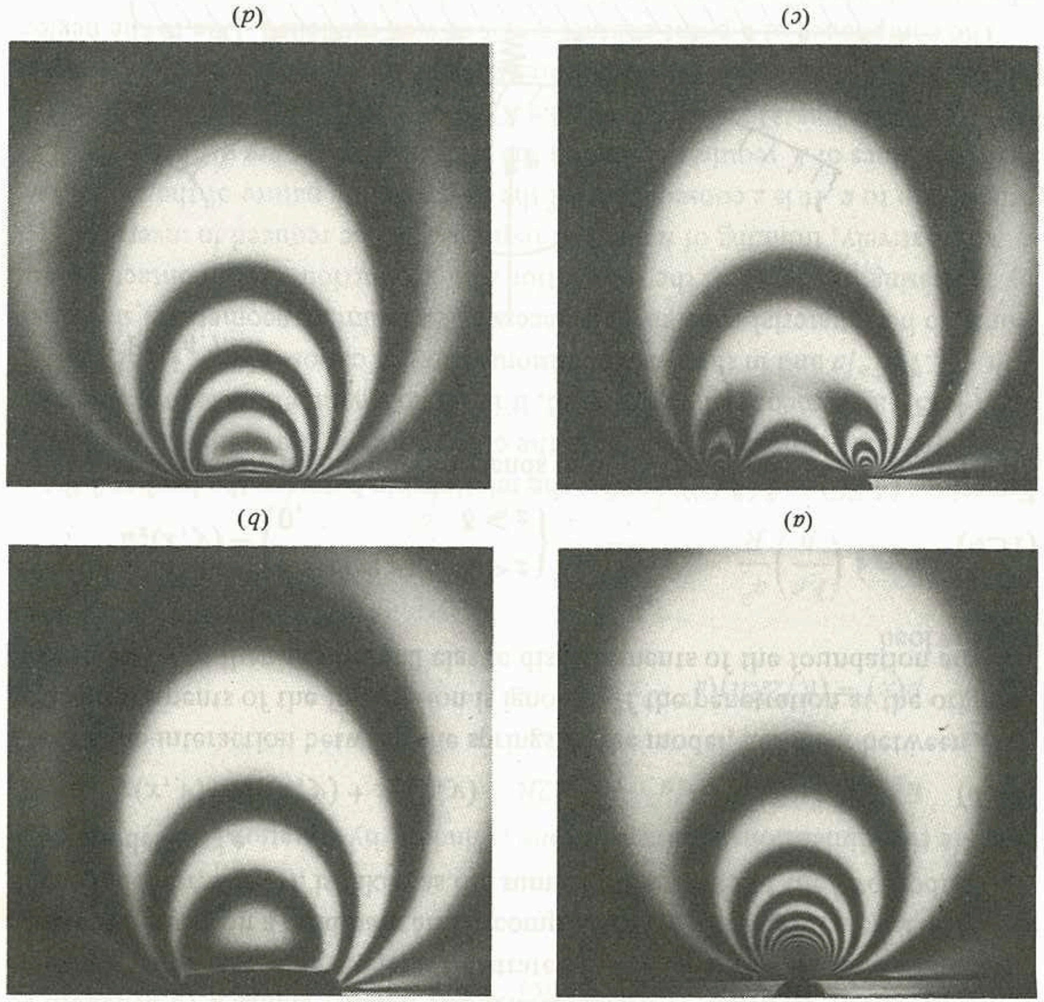
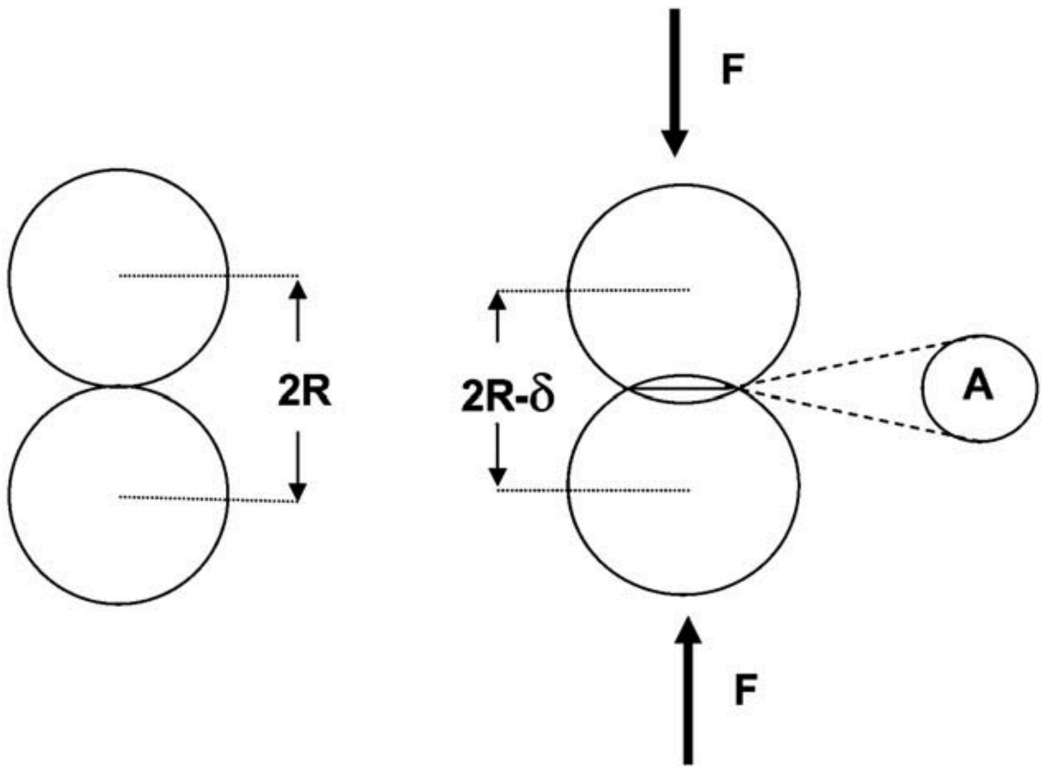


Fig. 4.6. Two-dimensional photo-elastic fringe patterns (contours of principal shear stress): (a) point load (§2.2); (b) uniform pressure (§2.5(a)); (c) rigid flat punch (§2.8); (d) contact of cylinders (§4.2(c)).

Hertzian contact law



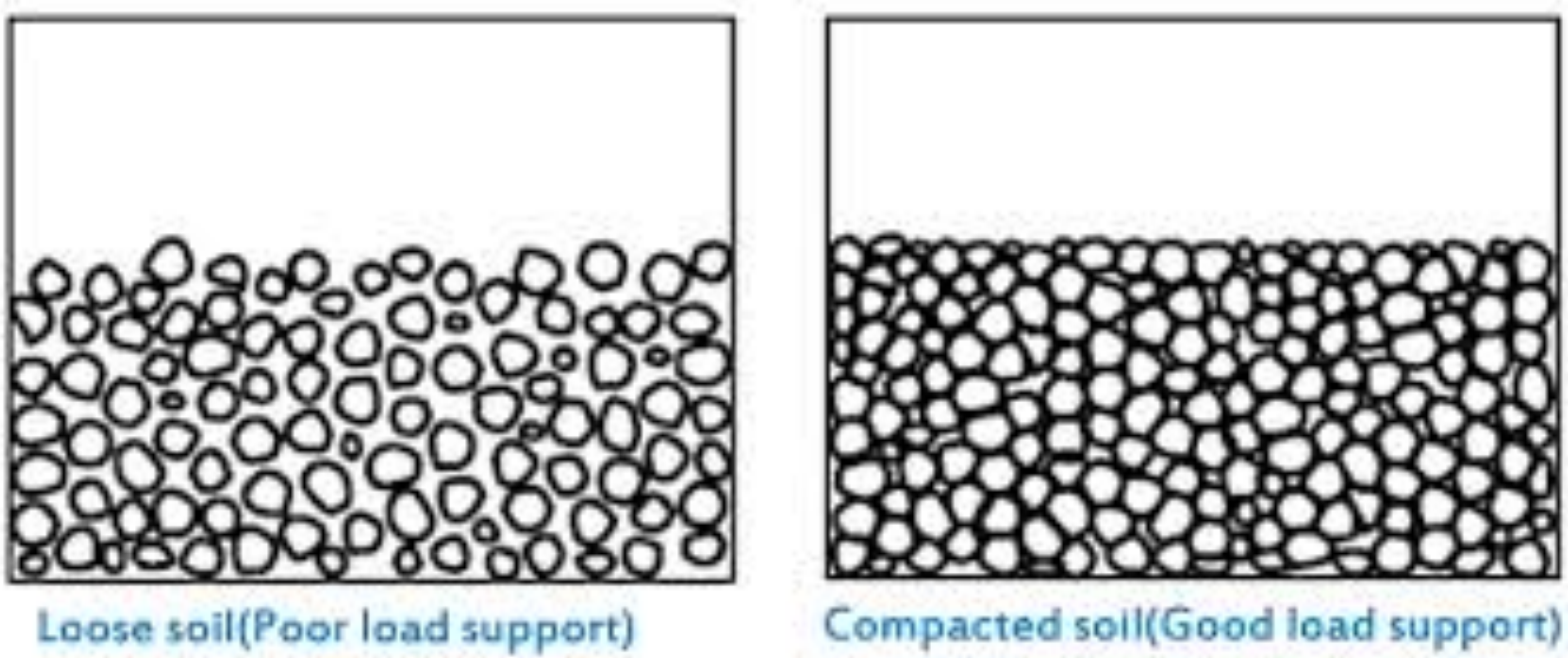
Contact stiffness

$$k = 6^{1/3} R^{1/3} \left(\frac{E}{1 - \nu^2} \right)^{2/3} F_n^{1/3}$$

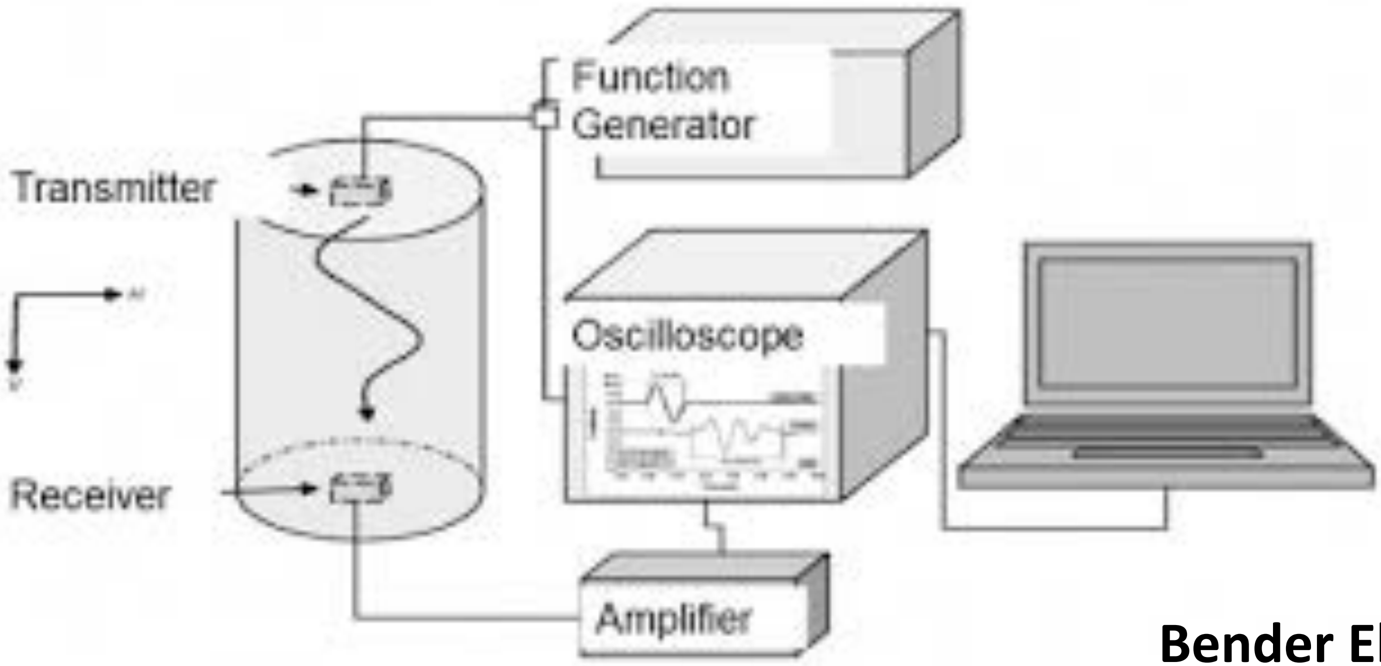
Solid state

Small strain (elastic) stiffness

Classical solids: elastic stiffness is a material constant
Granular materials: elastic stiffness depends on **pressure** and **volume fraction**

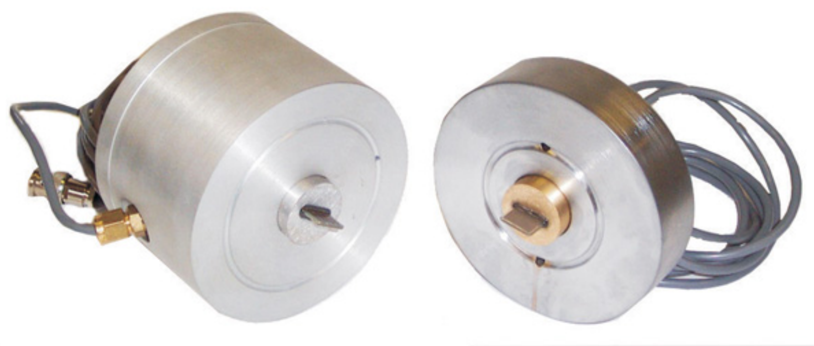


Small strain (elastic) stiffness

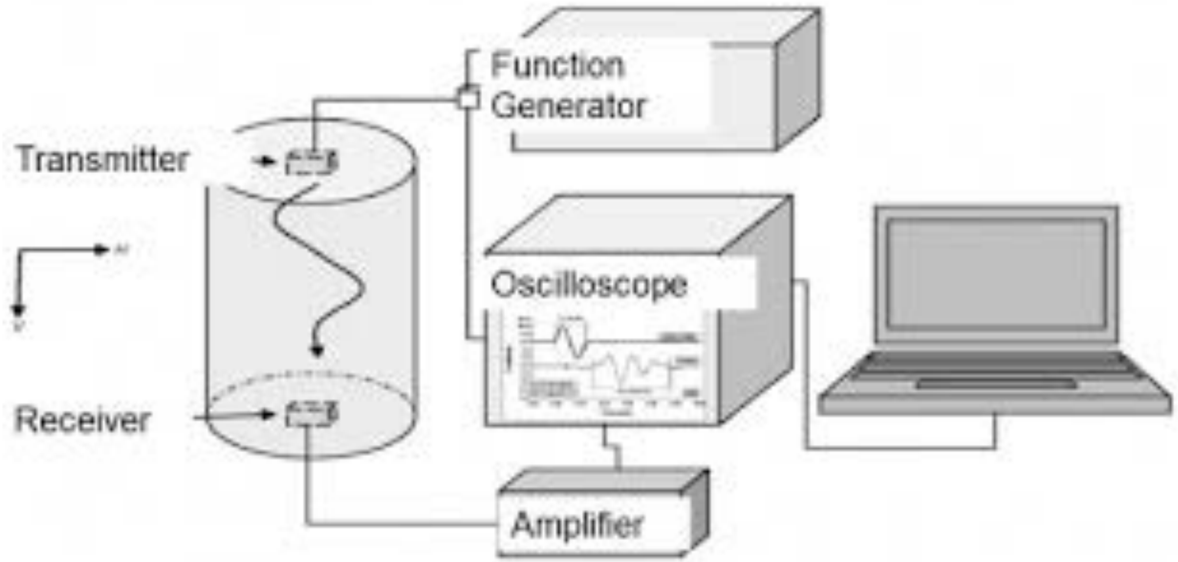


Cell with Bender Elements

Bender Elements



Small strain (elastic) stiffness



Soundspeed

$$V_p = V_p(\phi, p)$$
$$V_s = V_s(\phi, p)$$

Elastic moduli

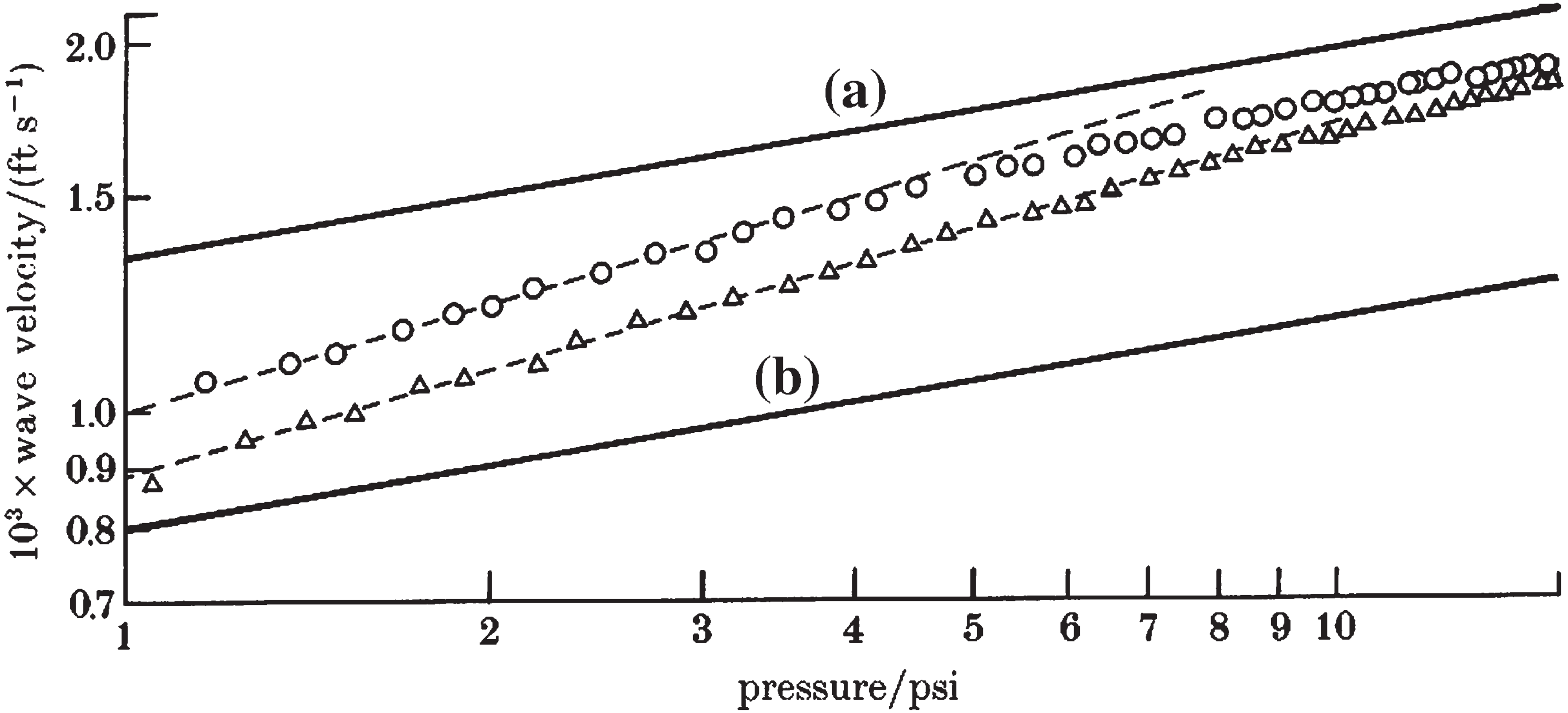
$$V_p = \sqrt{\frac{M_{bulk}}{\rho_{bulk}}}$$

$$V_s = \sqrt{\frac{G_{bulk}}{\rho_{bulk}}}$$

Small strain (elastic) stiffness

Soundspeed

$V_p = V_p(\nu, p)$
$V_s = V_s(\nu, p)$

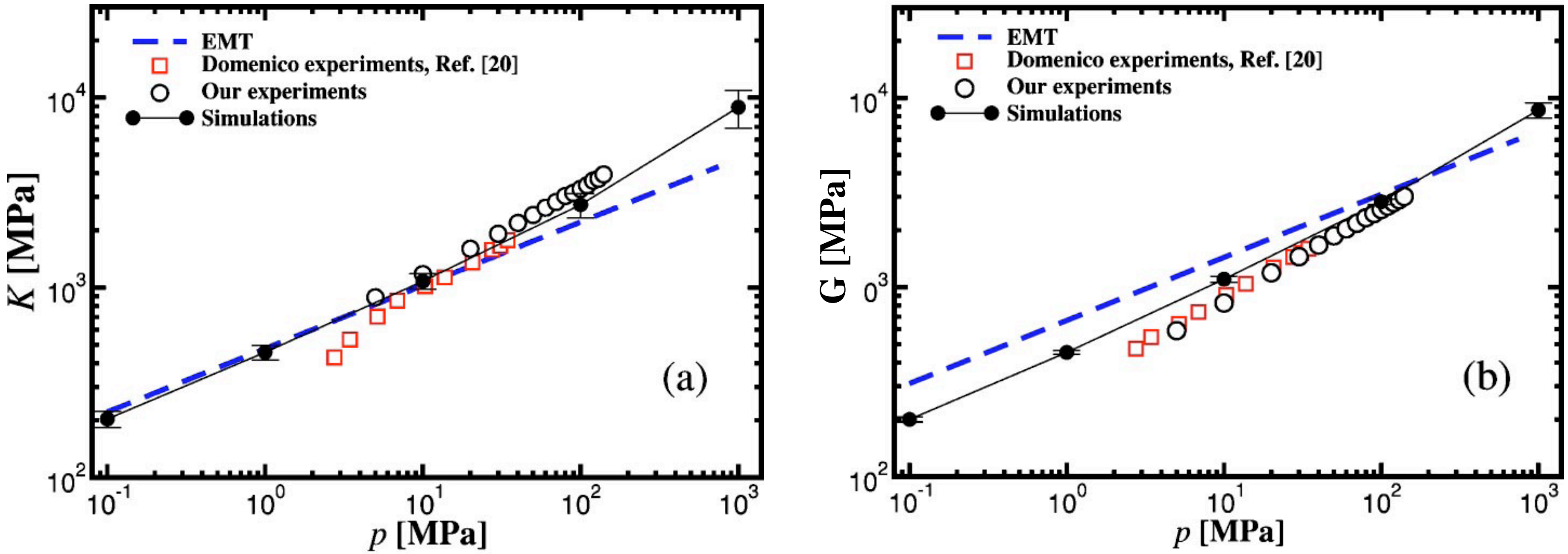


Small strain (elastic) stiffness

$$G_{bulk} \propto \frac{k}{R}$$

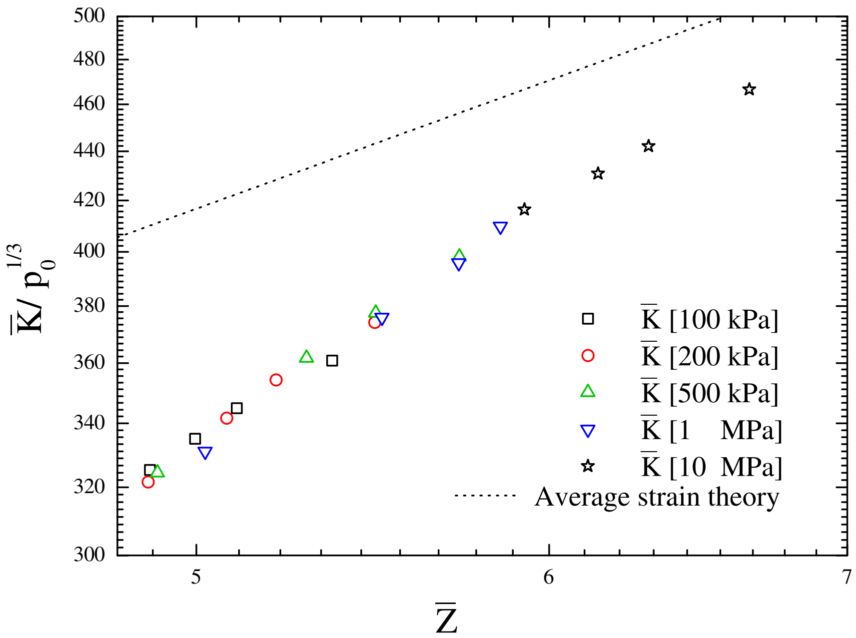
[Bathurst and Rothenburg, J. Appl. Mech. (1988)]

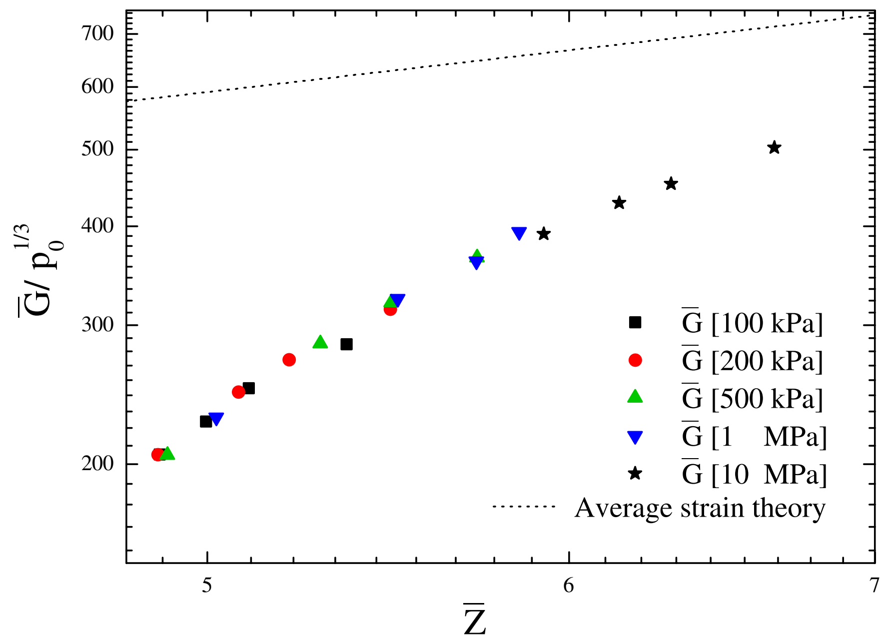
Because of Hertzian interaction we expect: $K(p) \propto G(p) \propto p^{1/3}$



[Gland et al., PRE (2005)]

Dependence on coordination number





Coordination number

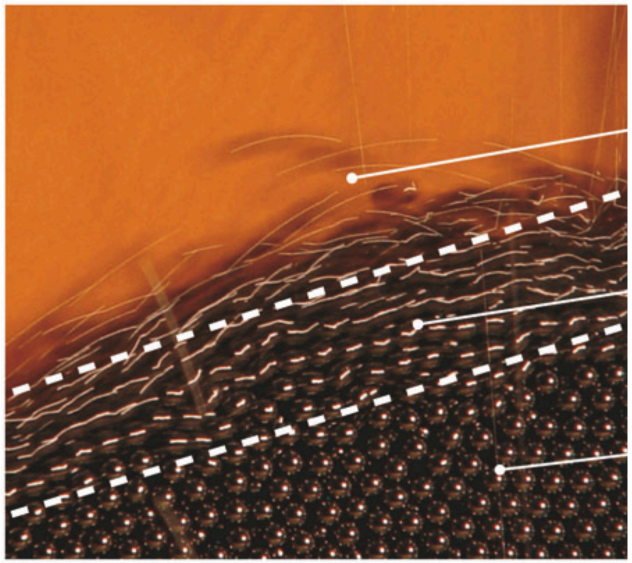
Avarage number of contacts in the system

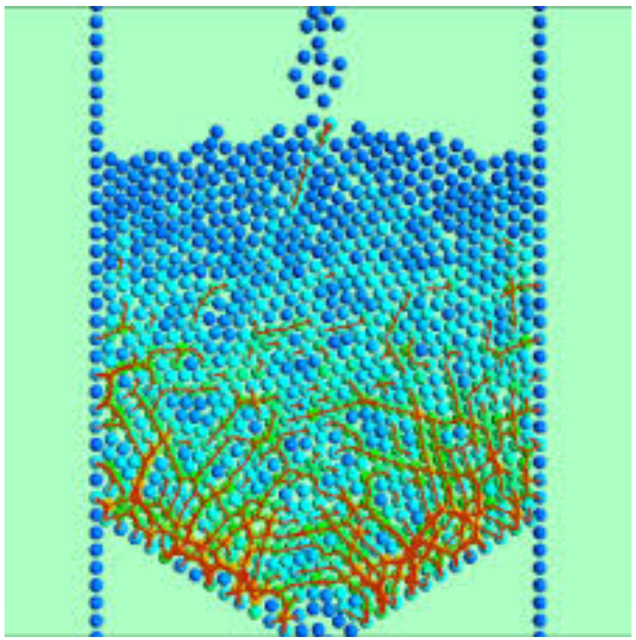
$$\bar{Z} = \frac{2Nc}{Np}$$

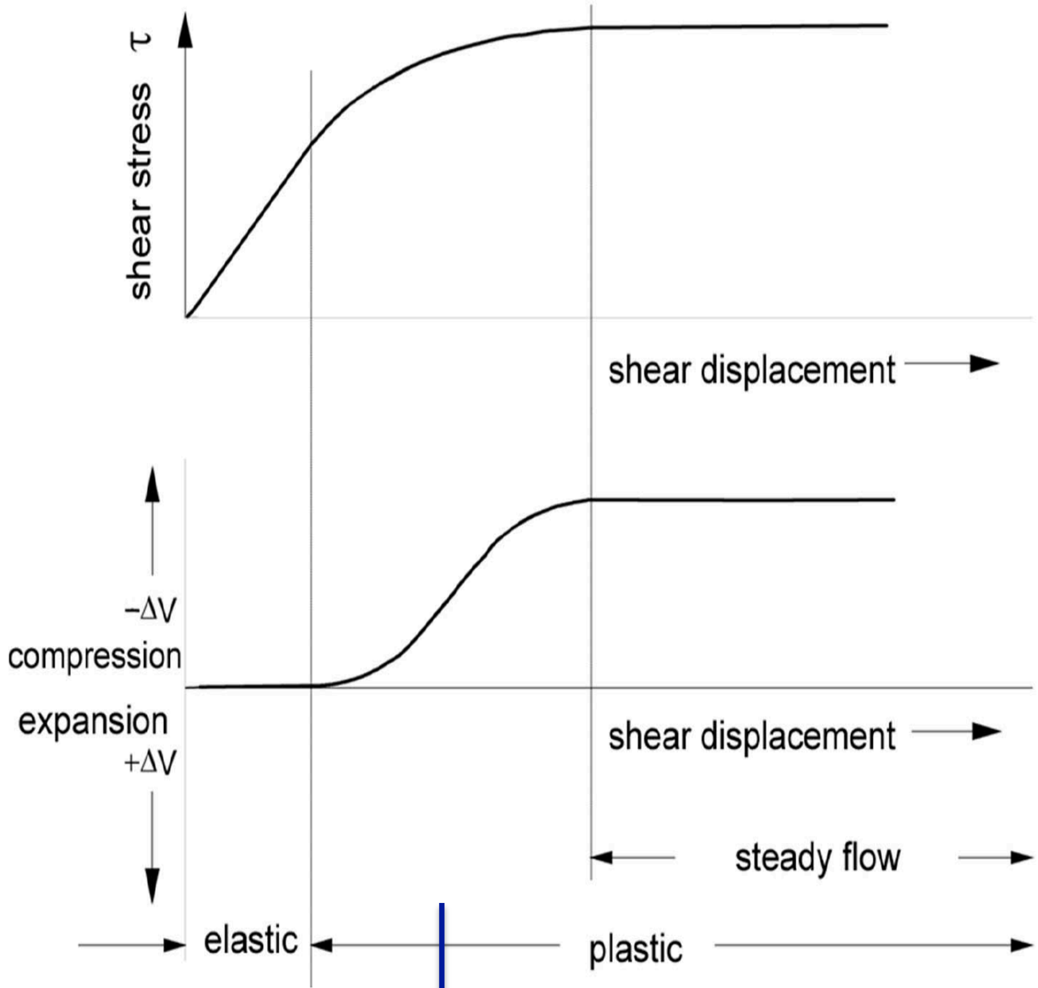
[Magnanimo et al., EPL (2008)]

Quasitatic behavior and flow threshold

Shearing







elsto-plasticity
hypoplasticity
…

Quasistatic behavior

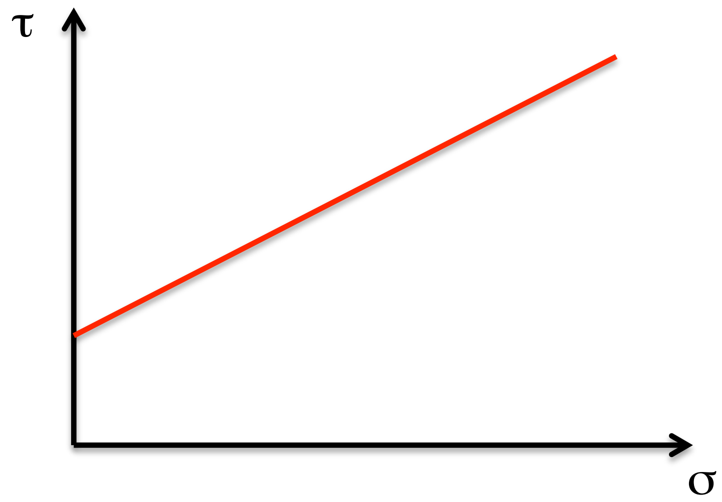
Coulomb (1773)
Yielding of granular material as frictional process
Interested in prediction of soil failures for Civil Engineering

$$\tau < c + \sigma \tan\phi$$

When $\tau = c + \sigma \tan\phi$ the material yields and starts to flow

c = cohesion
ϕ = friction angle

ϕ and c are material **constant**



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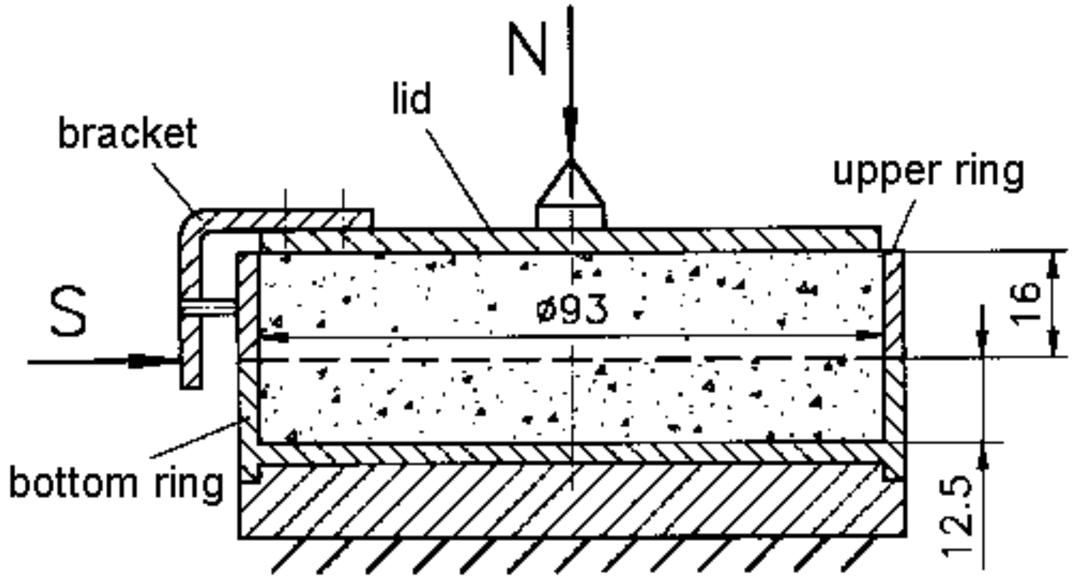
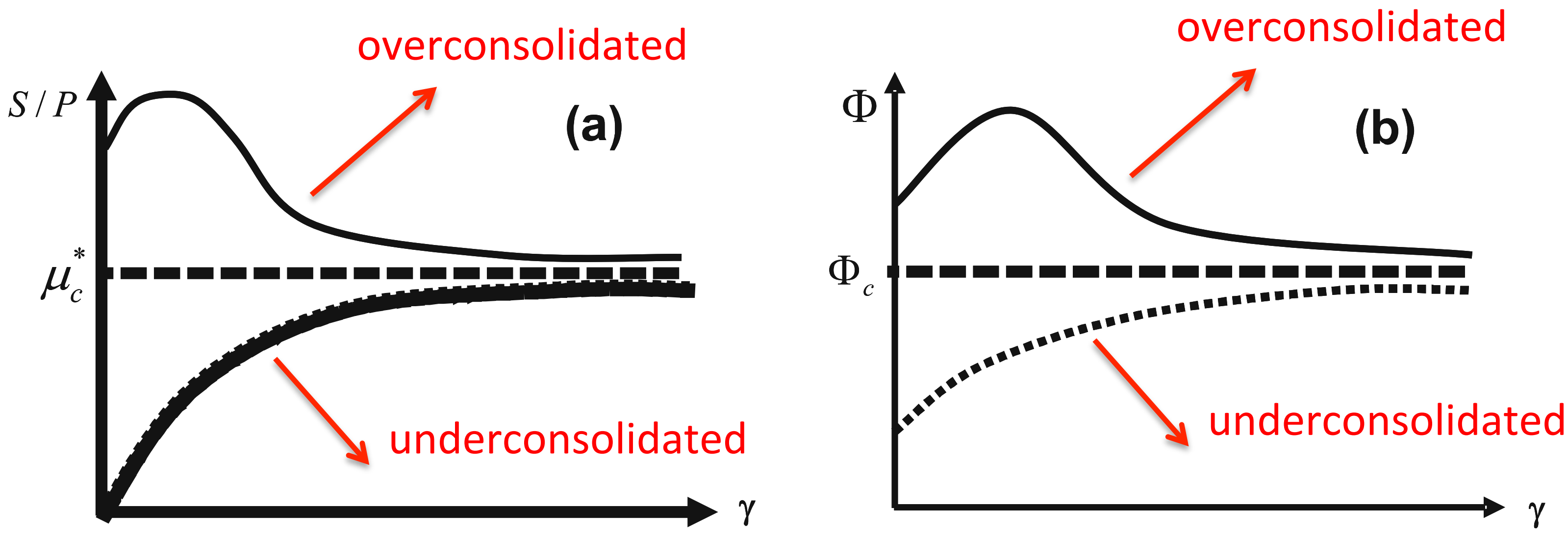


Critical state

A shearing granular material will ALWAYS approach a **critical** concentration
This is the **ONSET OF FLOW**

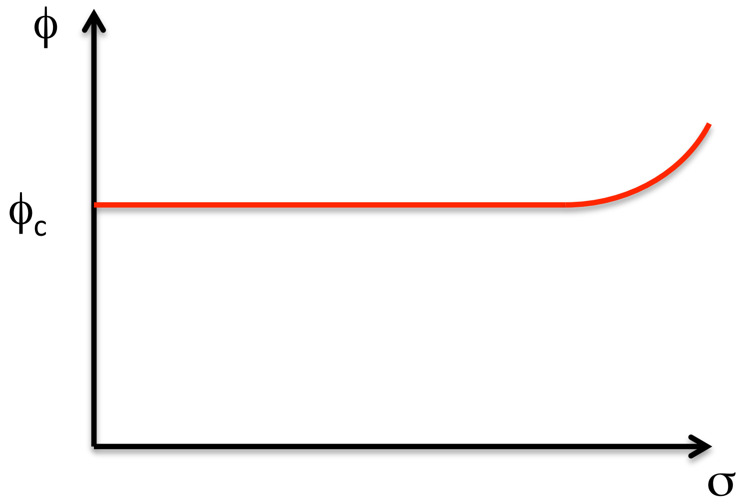


ϕ_c is again a material **constant**

The granular material **DILATES**

Critical state

Variation of critical concentration with applied stress



Most flows are characterized by low stress and large applied strain
→ we can assume incompressible flow at the critical concentration ϕ_c

Critical state

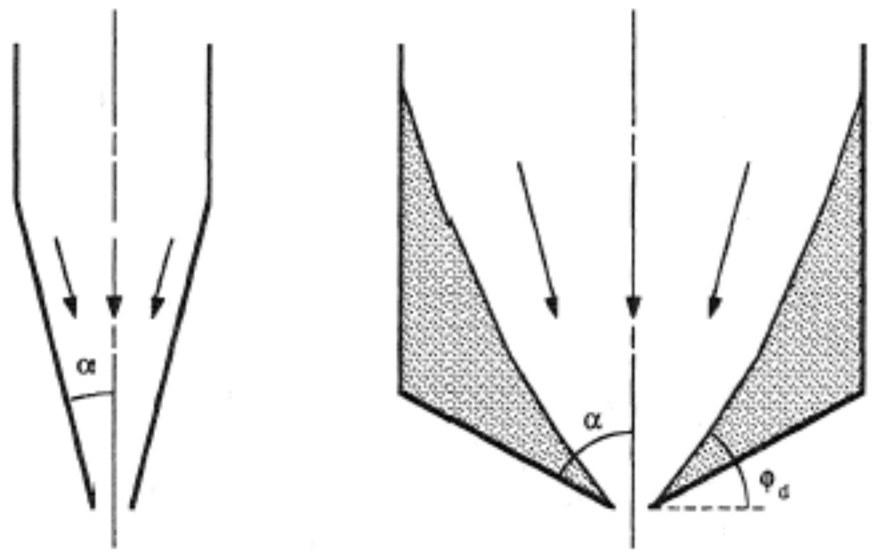
Soil mechanics: widely used

Particle Technology: flow behavior from silo (→ A. Kwade)

- when the material starts flowing is **always** yielding **everywhere** in the hopper (mass flow) or in a region (core flow)

$$\tau = c + \sigma \tan \phi$$

- the material is **always** at the critical concentration and it is **incompressible**.



Critical state

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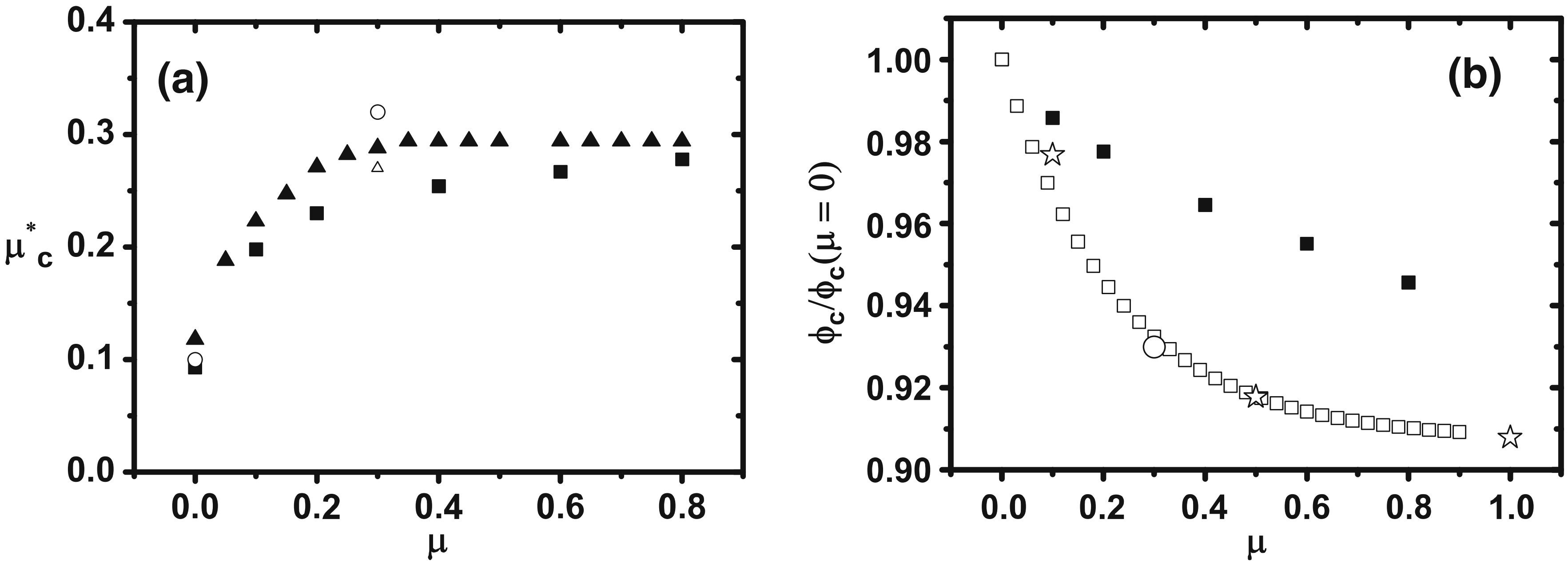
N.B.!!

Application of Critical State theory on is based **on Janssen theory:**
the pressure at bottom of the silo is independent of bed height

→ the whole bulk material is in the critical state.

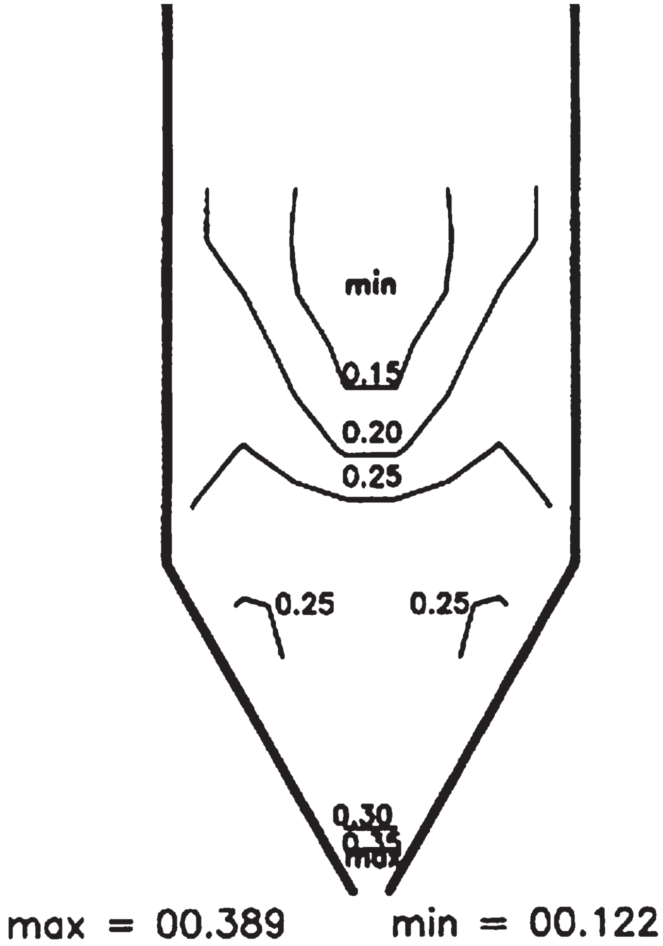
Dependence on microscopic properties



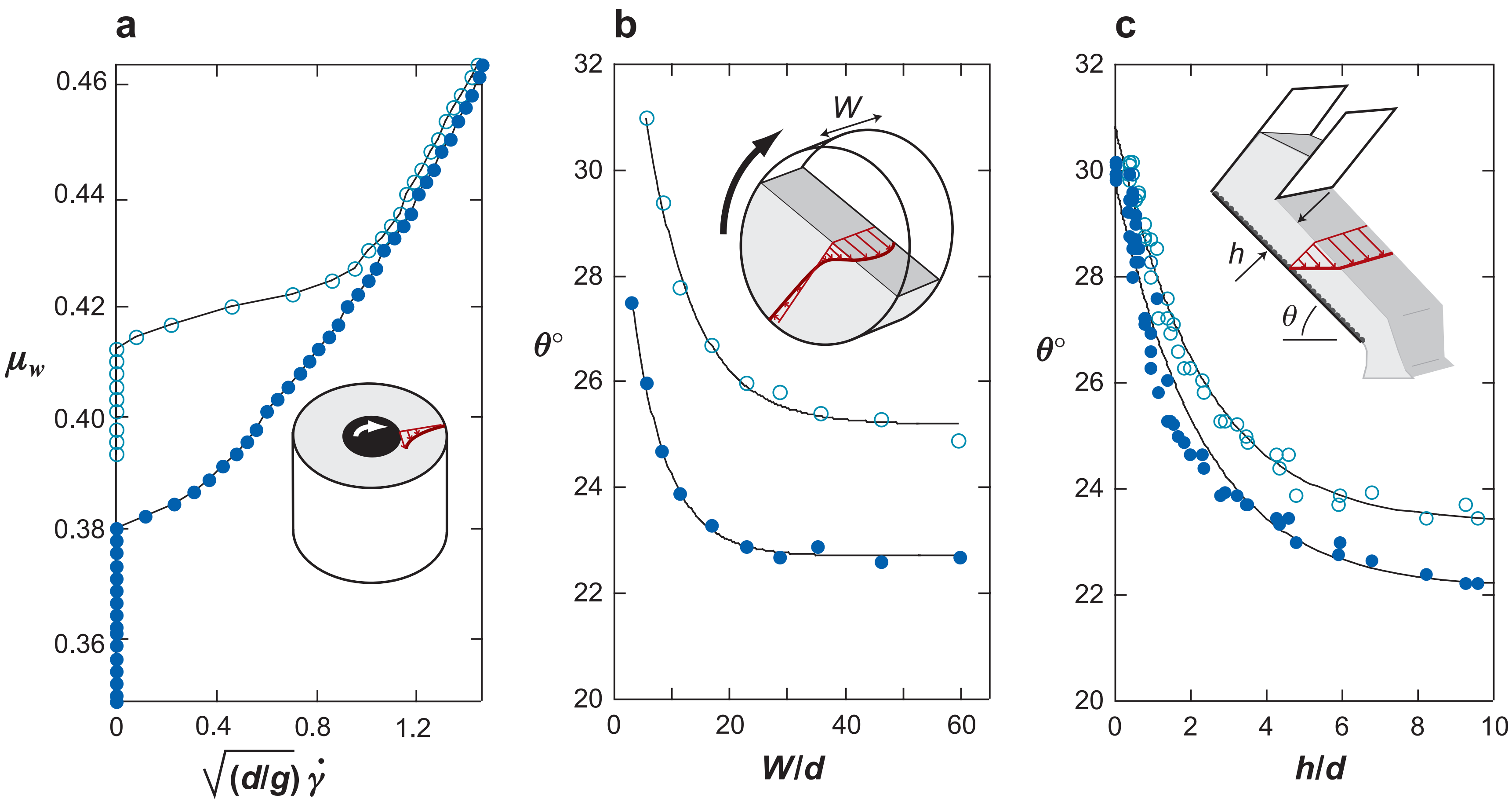
[Lamaitre et al., Rheol. Acta (2009)]

Critical state for silos - problems

ϕ is not constant in the silo

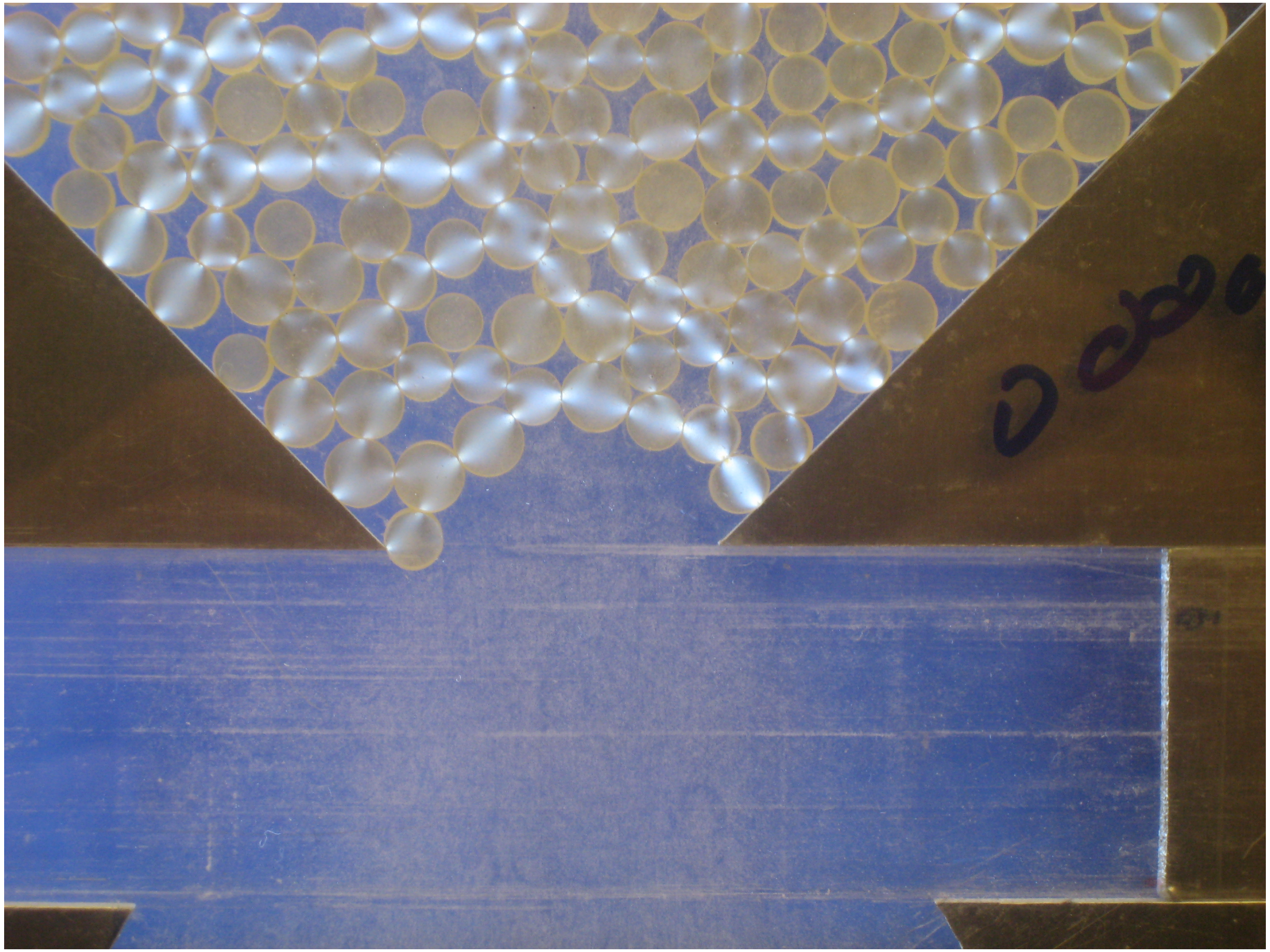


Critical state – can not describe hysteresis



Friction and dilatancy laws

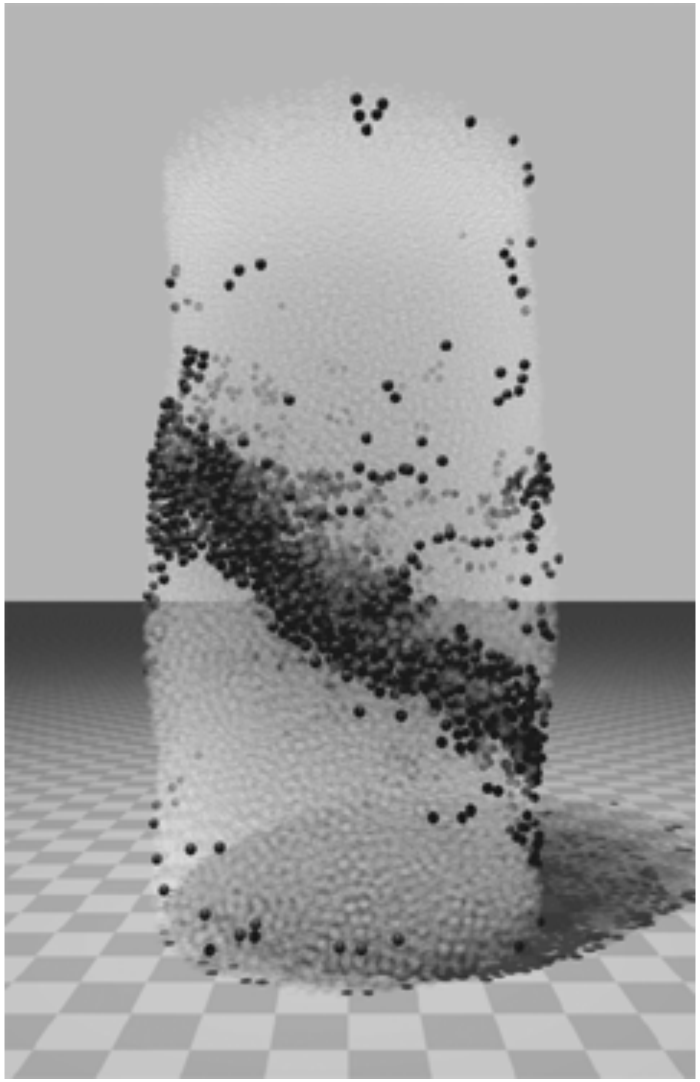
In solid and quasistatic flow, forces are transmitted through **force chains**



Shear bands and dilatant zones

Frictional Behavior = Mohr-Coulomb failure

Granular materials fail along narrow but **finite** zones: **SHEAR BANDS**

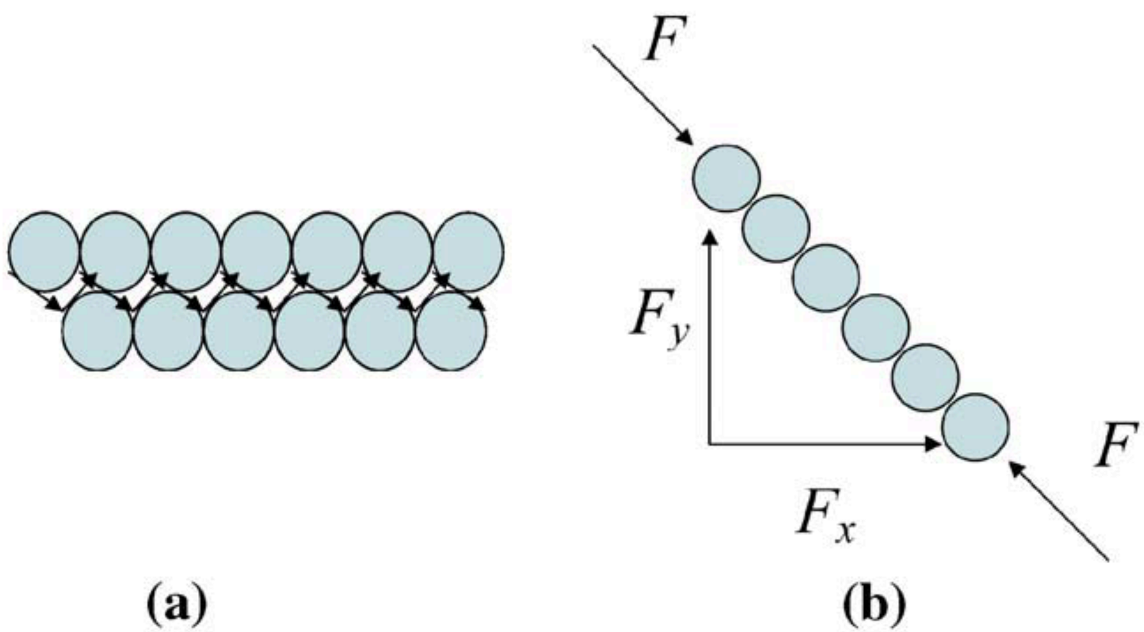


Shear bands and dilatant zones

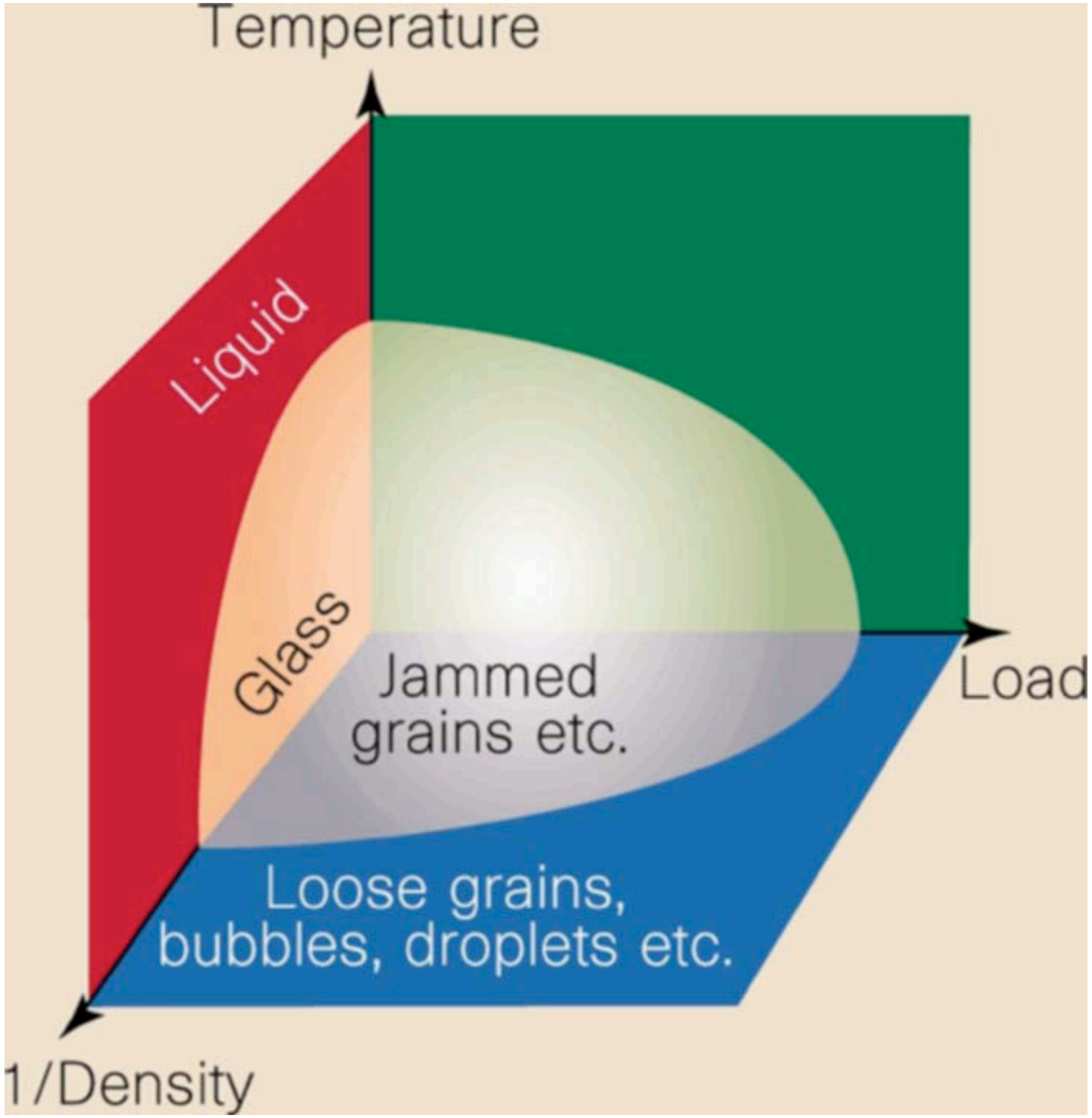
Frictional Behavior = Mohr-Coulomb failure
Granular materials fail along narrow but **finite** zones: **SHEAR BANDS**

GLOBALLY → frictional behavior
LOCALLY → force chains

$$\frac{\tau_{xy}}{\tau_{yy}} = \frac{\langle F_x l_y \rangle}{\langle F_y l_y \rangle} = const$$



Jamming phase diagram



[Liu and Nagel., Nature (1998)]

Jamming robot

[Brown et al., PNAS (2010)]

Collisional or rapid granular flows

Dimensionless analysis (Buckingham Pi theorem)

Bagnold theory

$$\sigma_{ij} = f(\phi, \rho_p, d, \dot{\gamma})$$

$$\sigma_{ij} = f(\phi)\rho_p d^2 \dot{\gamma}^2$$

The shear stress varies as the square of the shear rate

Granular Temperature

1. granules moving in a flow = molecules in the kinetic theory of gases
2. random velocities = thermal motion of molecules.

Granular temperature = magnitude of fluctuating velocities

$$T_g = \frac{1}{3}\left|\left\langle u'^2_i \right\rangle\right| = \frac{1}{3}\left(\left\langle u'^2 \right\rangle + \left\langle v'^2 \right\rangle + \left\langle w'^2 \right\rangle\right)$$

Trace of the streaming stress tensor

$$T_g = \frac{1}{3\rho\phi} tr(\sigma^s_{ij})$$

deriving a set of equations for **Rapid Granular Flows**

Granular Hydrodynamic

Conservation of mass

$$\frac{D\rho\phi}{Dt} = \rho\phi\nabla\cdot\underline{u} = \mathbf{0}$$

Conservation of momentum

$$\rho\phi\frac{D\underline{u}}{Dt} = \nabla p(p,\phi,T_g,e) + \nabla\cdot\left(\eta(\rho,\phi,T_g,e)\nabla\underline{u}\right)$$

Conservation of granular energy (granular temperature)

$$\rho\phi\frac{DT_g}{Dt} = \nabla\cdot\left(\alpha(\rho,\phi,T_g,e)\nabla T_g\right) + \underline{\sigma}\cdot\nabla\underline{u} - \Gamma(\rho,\phi,T_g,e)$$

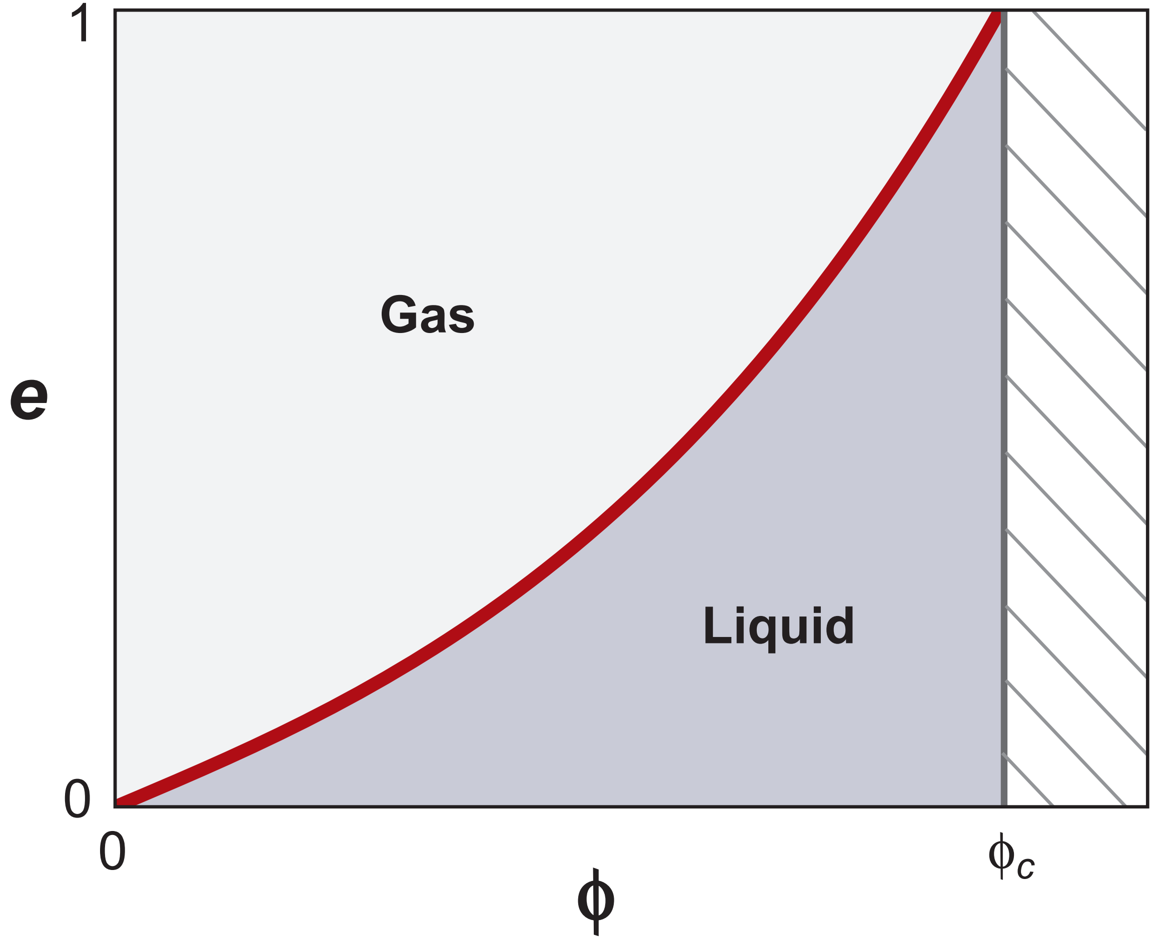
Kinetic Theory – Range of applicability

- **Nearly elastic particles** (e=0.9)
- **Extremely small concentration:**
  magnitude of thermal velocities is much larger than the relative velocities induced by shear

Binary collisions

- **Isotropy in the angular distribution of collisions**
- **Molecular chaos:**
  no correlations in the velocities or positions of colliding particles
- **Absence of friction between particles and walls:**
  silos can not be modeled with kinetic theory

Kinetic Theory – Range of applicability



[Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Kinetic Theory – Extended theories

Jenkins, *Dense shearing flows of inelastic disks.* Phys.Fluids (2006)

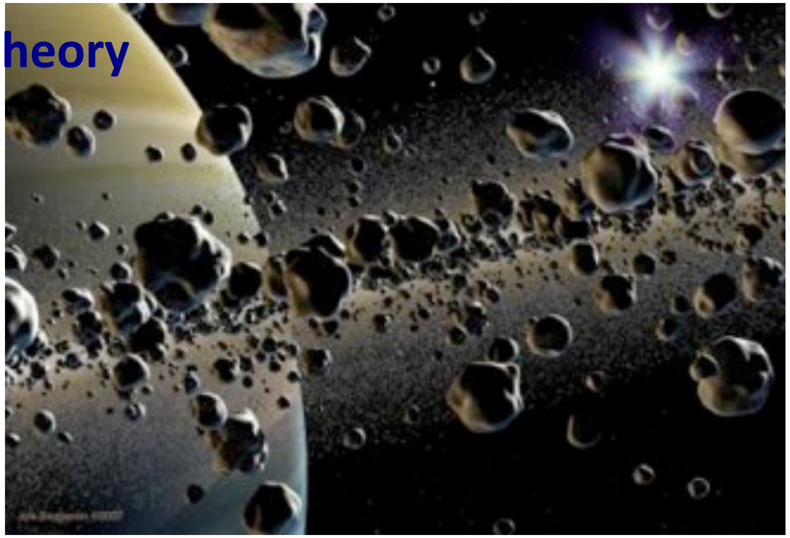
Vescovi, Di Prisco & Berzi, *From solid to granular gases: the steady state for granular materials* (2013)

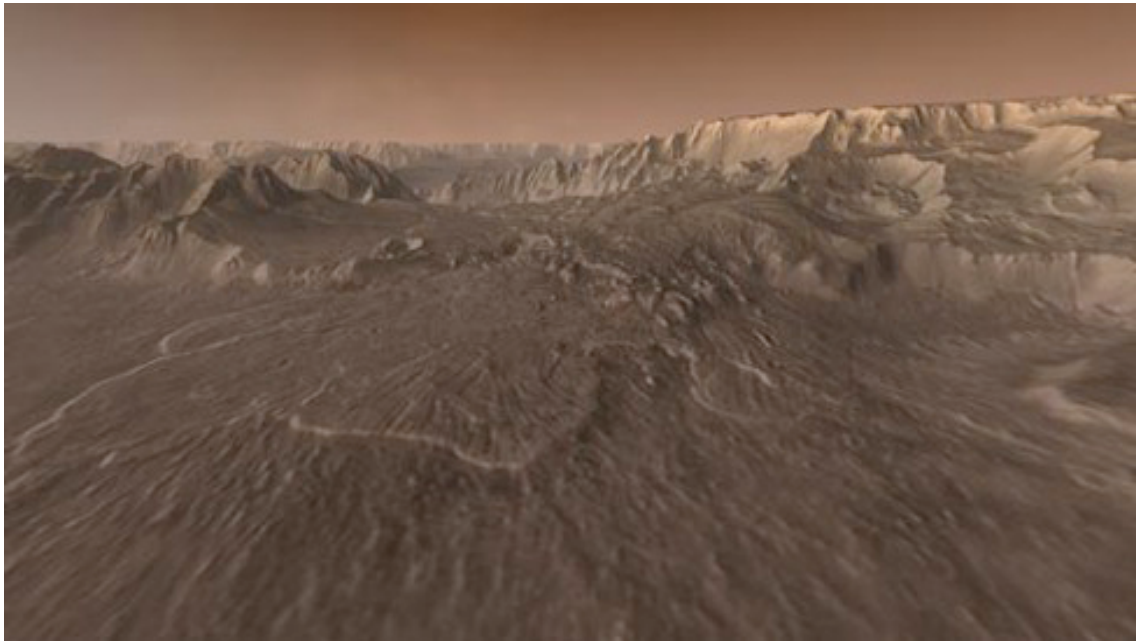
[…]

Granular material: continuum approach

Solid: soil mechanics

Gas: kinetic theory

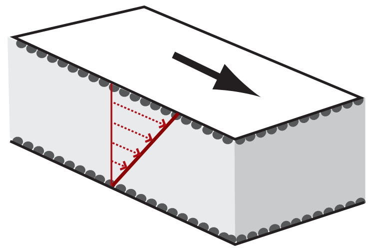




Liquid ??

Dense (slow) flows
and inertial regime

Inertia number



For **large** systems – and **rigid** grains

Only based on dimensional analysis
The transition can be described trough a **single dimensionless number**

$$I = \frac{\dot{\gamma} d}{\sqrt{p/\rho_p}}$$

Inertia number

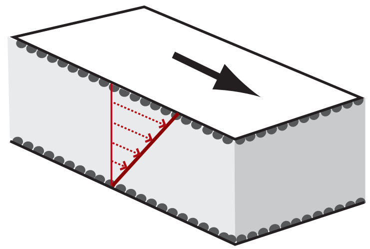
quasi-static (solid) dense (liquid) rapid (gas)

$\longrightarrow I$

10^{-3} 10^{-1}

[Da Cruz et al. (2005); Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Inertia number



$$I = \frac{\dot{\gamma} d}{\sqrt{P / \rho_p}}$$

I = micro time scale / macro time scale

$$\frac{d}{\sqrt{P / \rho_p}}$$

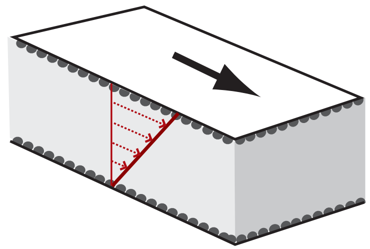
microscopic time scale
time needed for a particle to fall in a hole of size *d*
under the pressure *P*
- typical time scale of rearrangements -

$$\frac{1}{\dot{\gamma}}$$

macroscopic time scale
linked to the mean deformation

[Da Cruz et al. (2005); Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Inertia number



$$I = \frac{\dot{\gamma} d}{\sqrt{\cancel{p} / \cancel{\rho_p}}}$$

I = micro time scale / macro time scale

I **small –** quasi-static
macroscopic deformation is slow compared to microscopic rearrangement

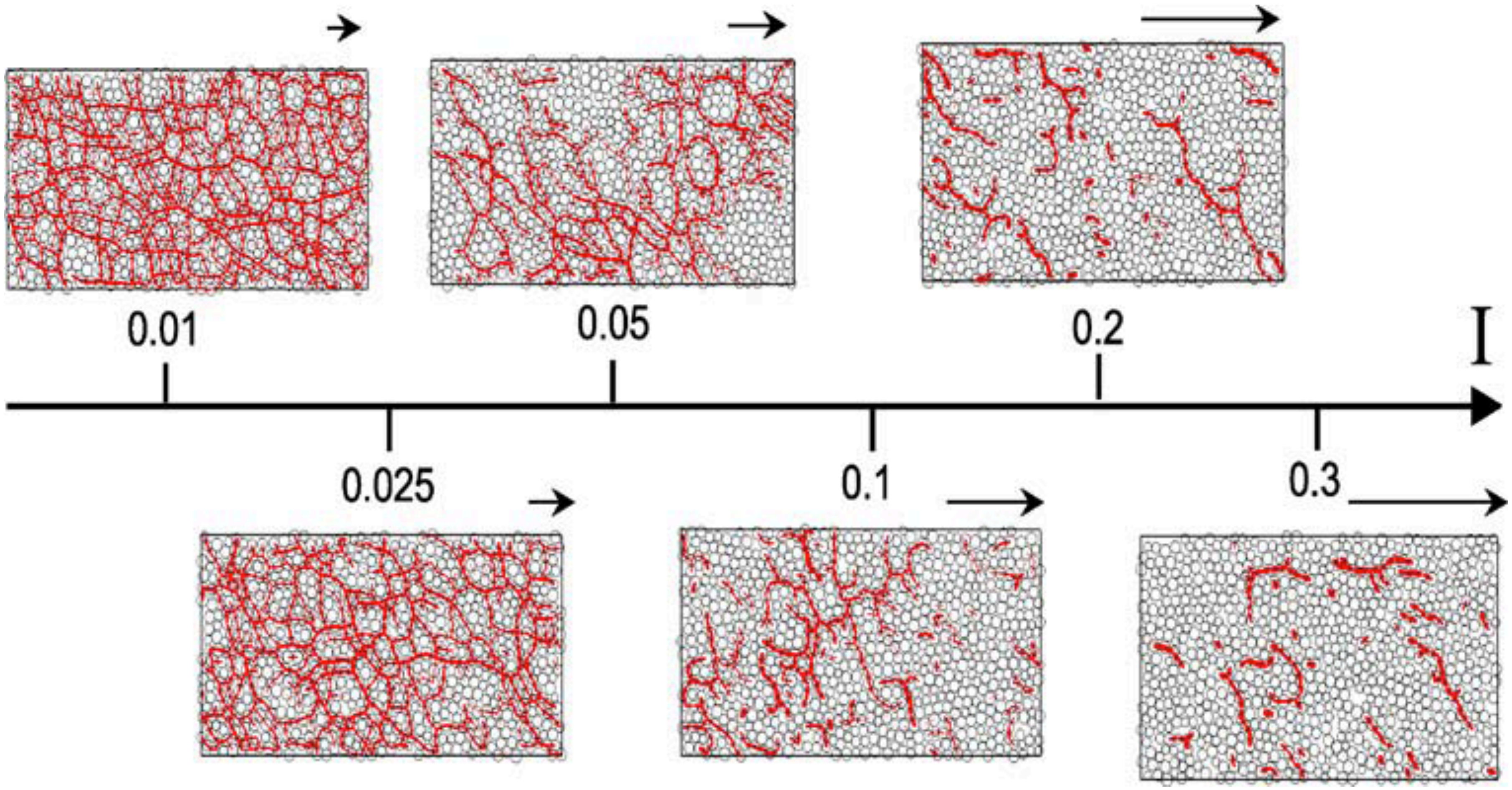
I **large –** rapid flows

quasi-static (solid) dense (liquid) rapid (gas)

$$\xrightarrow{\quad 10^{-3} \qquad\qquad 10^{-1} \quad} I$$

[Da Cruz et al. (2005); Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

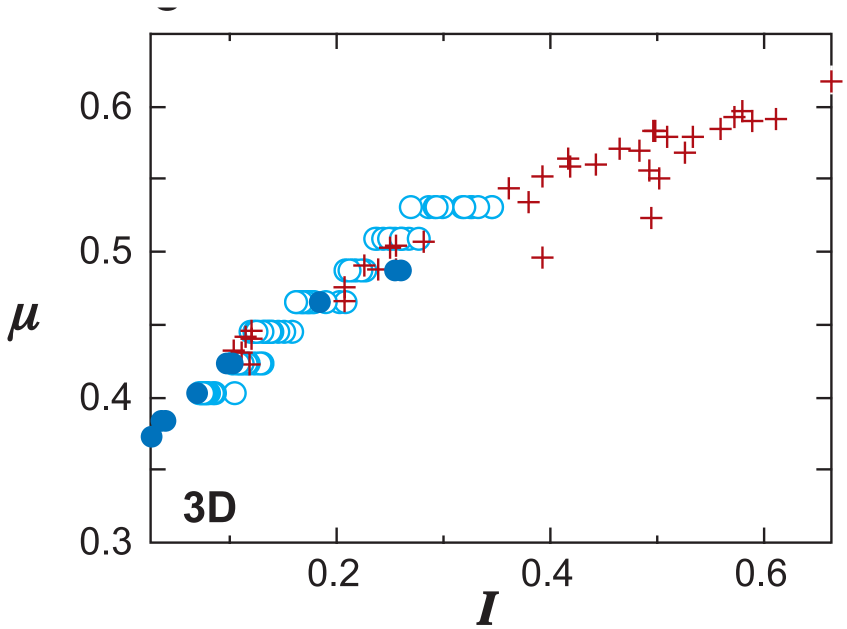
Quasistatic → Dense → Rapid

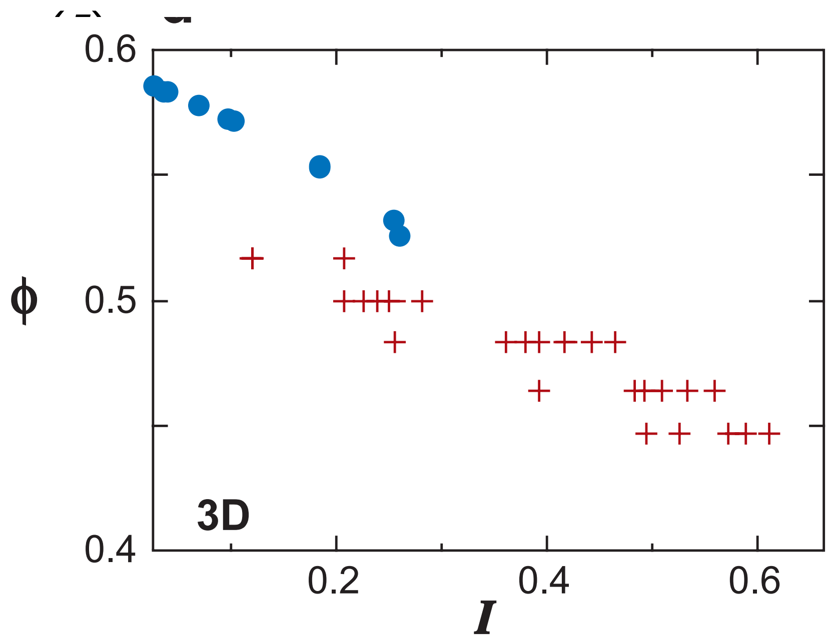


Pouliquen μ-I rheology

$$\tau = \sigma\mu(I) \qquad \phi = \phi(I)$$

(**rigid** grains)





- inclined plane (exp, num)
- annular shear (exp)

glass beads : $\mu_1 = \tan 21^\circ$

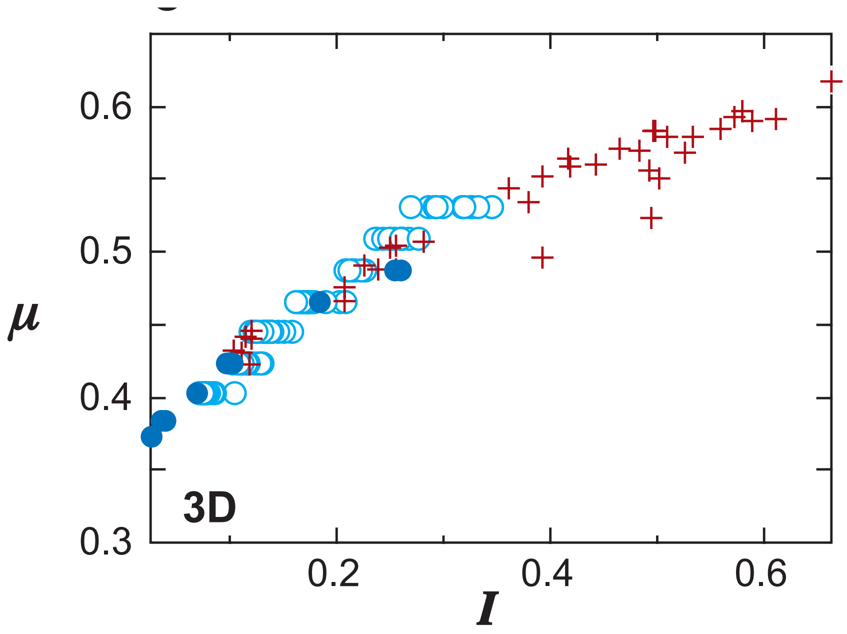
$\mu_2 = \tan 33^\circ \qquad I_0 = 0.3$

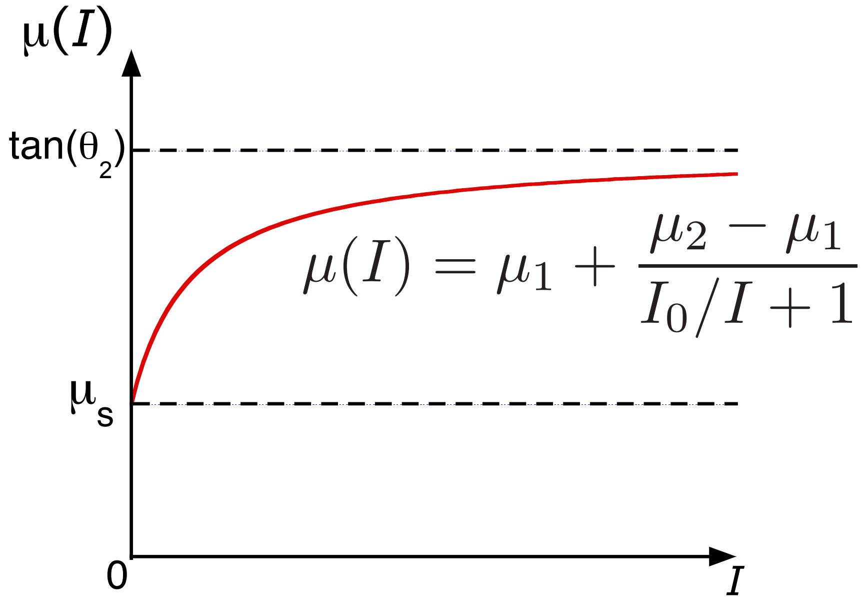
[Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Pouliquen μ-I rheology – local constitutive relation

$$\tau = \sigma\mu(I) \qquad \phi = \phi(I)$$

(**rigid** grains)





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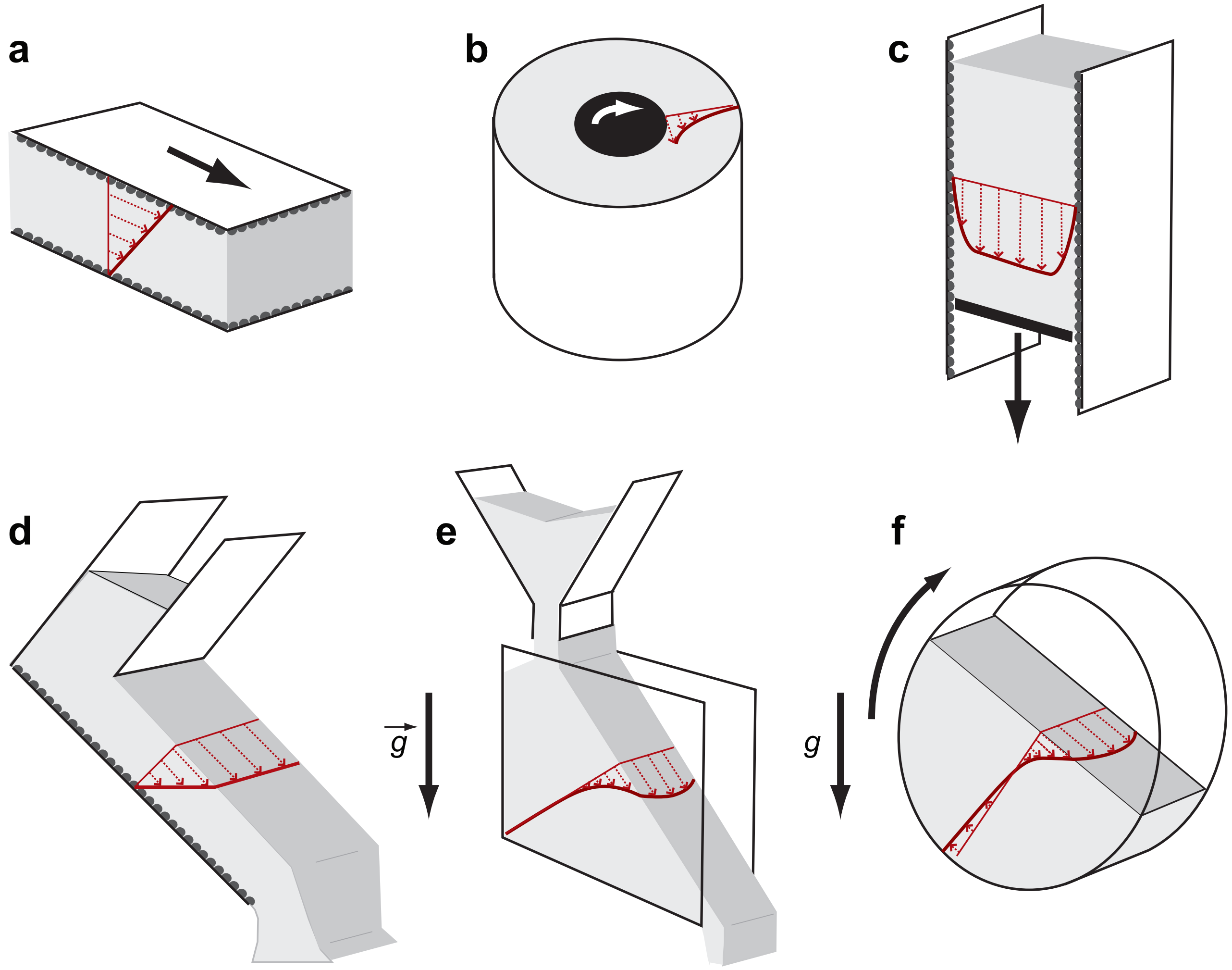
$\mu_2 = \tan 33^\circ \qquad I_0 = 0.3$

[Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Different geometries



[Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008), Weinhart et al. Phy. Fluids (2013)]

Pouliquen μ-I rheology – Tensorial extension

1) incompressible media, no normal stress difference

$$\boxed{\sigma_{ij} = -P\delta_{ij} + \tau_{ij}}$$

2) $\dot{\gamma}_{ij}$ et τ_{ij} are colinear

$$\boxed{\tau_{ij} = \frac{\mu(I)P}{|\dot{\gamma}|}\dot{\gamma}_{ij}}$$

$$\dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

$$|\dot{\gamma}| = \sqrt{\frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$$

effective viscosity

(Drucker-Prager criterion)

Flow threshold $|\tau| = \mu_1 P$

$$|\tau| = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}}$$

[Jop et al. Nature (2006)]

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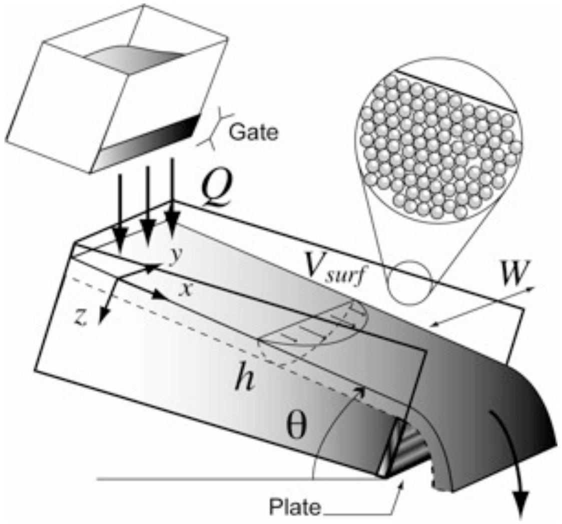
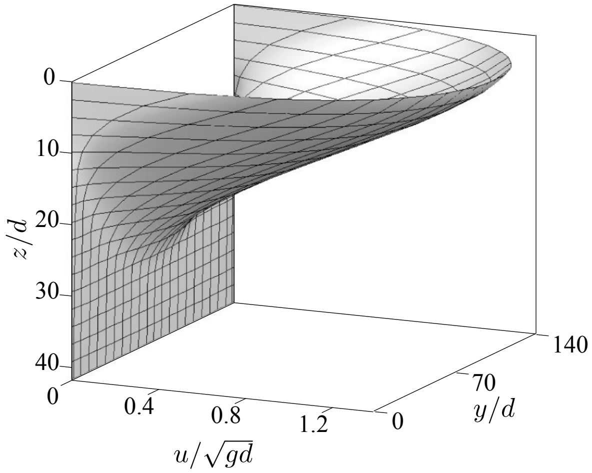
effective viscosity

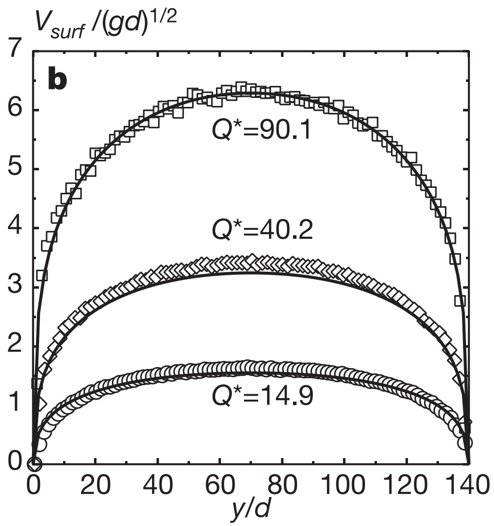
OK for: free surface flows between rough walls (Jop, 2006)
flows down an inclined plane (Forterre, 2006)

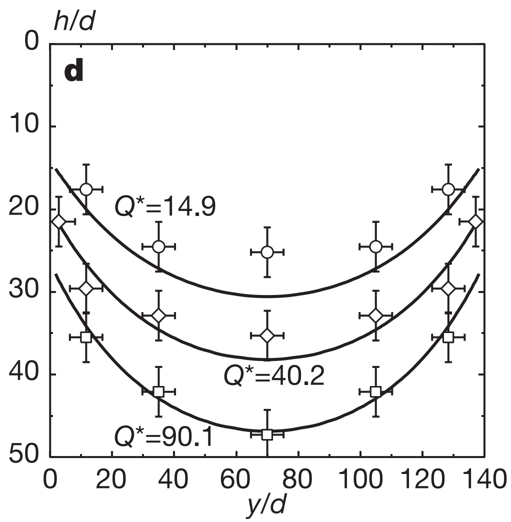
[Jop et al. Nature (2006)]

Test of the viscoplastic law :
1- Heap flows



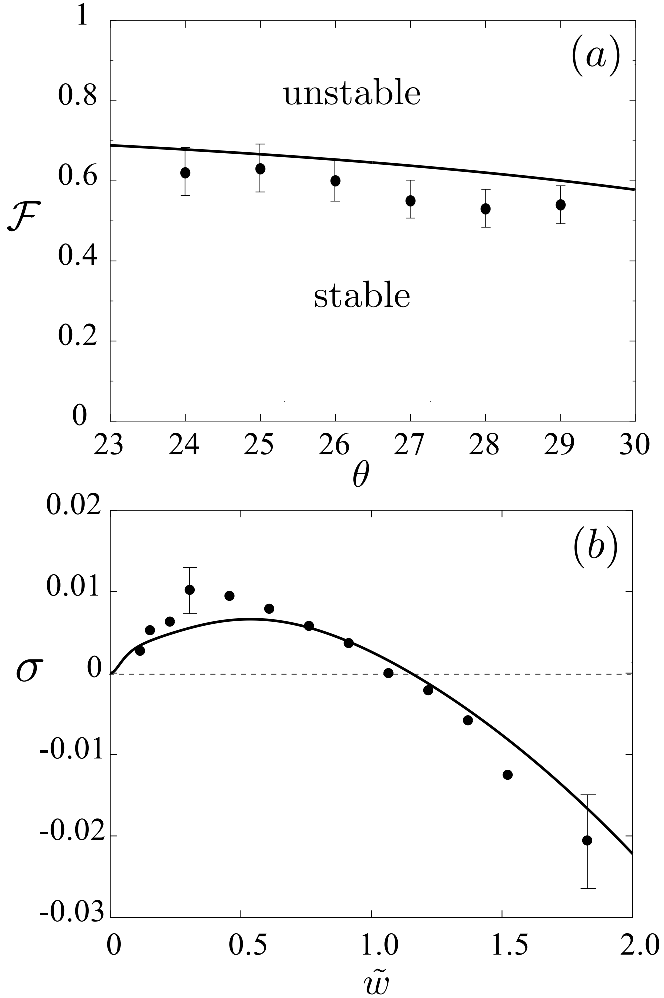






Jop et al. Nature 2006

Test of the viscoplastic law :
2 - Surface instability



Forterre J. Fluid Mech. 2006

Limits of μ-I rheology

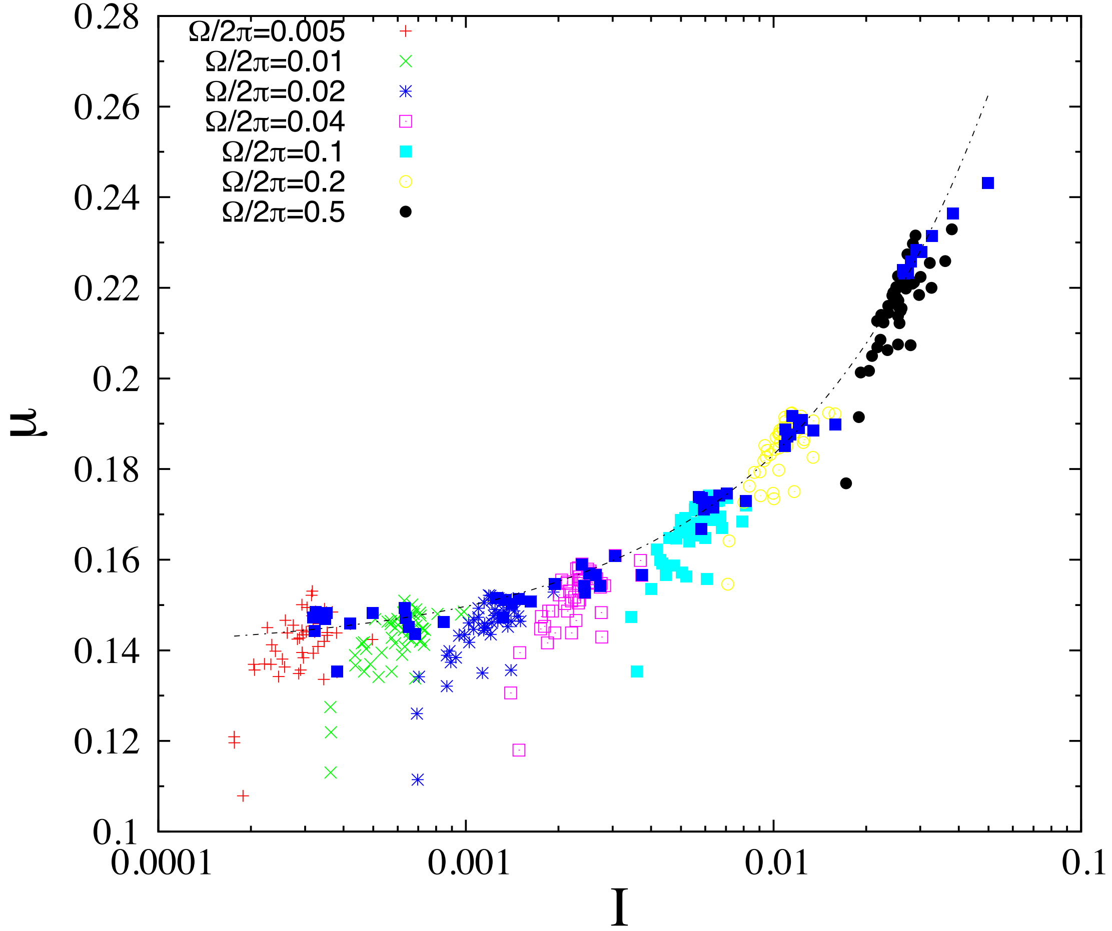
- **Microscopic origin**
  phenomenological – some work in [Weinhart et al. Phy. Fluids (2013)]

- **Transition to Quasi-static regime**
  shear bands not described
  Transition between rate-independent and rate-dependent regimes
  1 solution: **Non-local models**

- **Transition to Kinetic regime**
  collisional flows not correctly described
  modified kinetic theory by introducing a rate-independent term

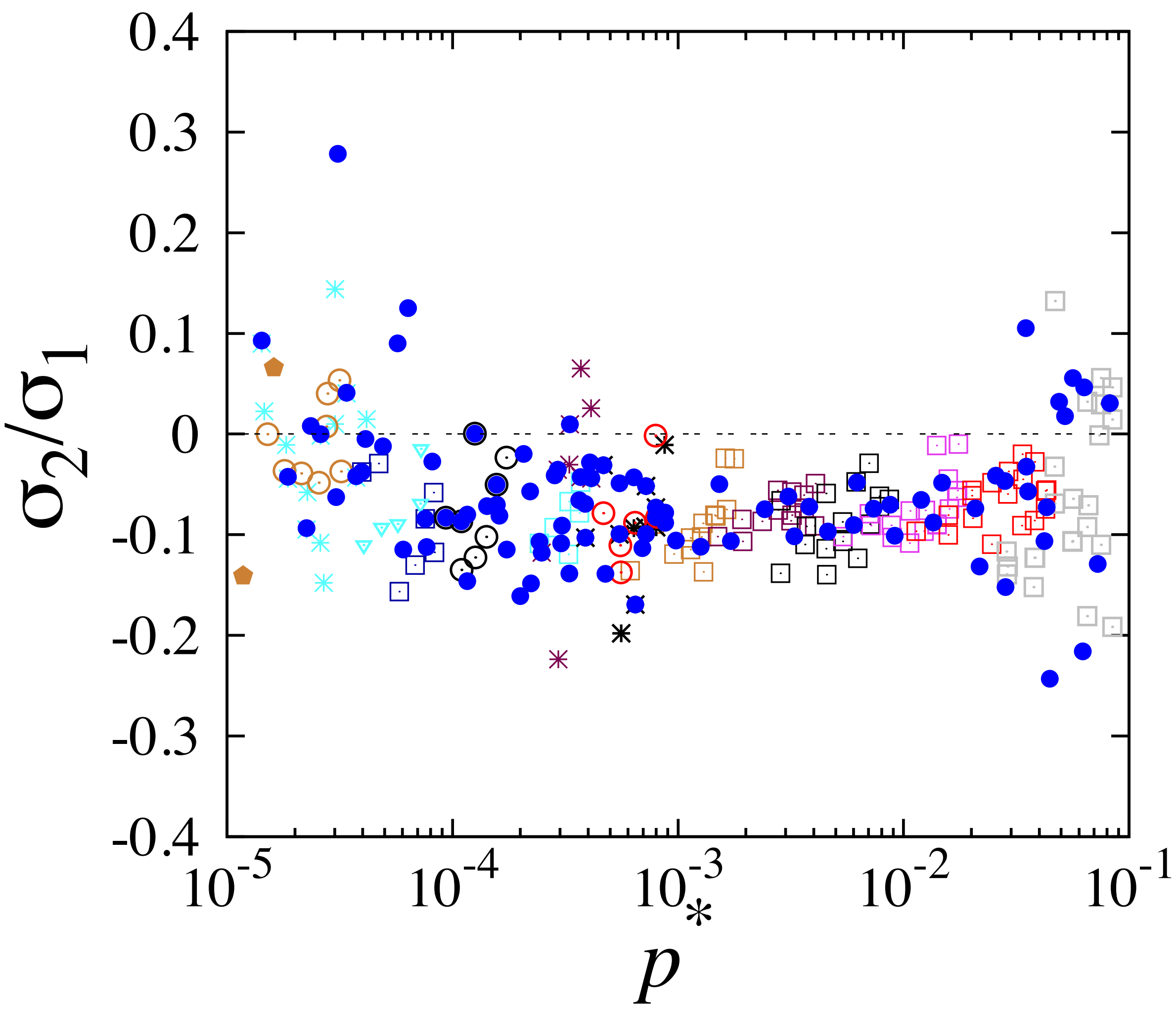
[Forterre and Pouliquen, Ann. Rev. Fluid Mech. (2008)]

Non local effects



[Singh et el., under review New J. Phys. (2015)]

Solid - fluid duality



[Singh et el., under review New J. Phys. (2015), Weinhart et al. Phy. Fluids (2013)]