Overview of segregation:
From inclined planes to drums; via a volcano

A. R. Thornton 2$^{\text{th}}$ May 2019

1. Introduction
2. Introduction to mixing
   - Type of mixers
3. Gravity driven flows
4. A continuum model of segregation
5. Multiscale modelling
6. Coarse-graining
7. Closing the model
8. Experimental, and simulations validation
9. Coupled Theory of Segregation
   - Granular fingering
   - One-dimensional travelling wave solution
   - Grid dependence
10. To rotating drums
    - Segregation in long rotating cylinders
11. Conclusions

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Granular segregation, hard or easy?

- Granular segregation is very easy to observe, preventing segregation is often the problem.
- Segregation in granular materials can occur for a number of reasons
  - Difference in size
  - Difference in size
  - Difference in density
  - Difference in contact properties
  - Difference in angle of repose
  - Differential forcing (air drag etc...)
  - plus many others ....

Extract from a 1978 paper : Particle segregation ... and what to do about it

Do's and don'ts of handling segregating materials

1. Do use mass-flow bins with tall cylindrical sections.
2. Do use a mixing device for charging a multiple-outlet bin at the center to cause uniform symmetric segregation.
3. Do proportion and mix badly segregating materials just before process usage, with as little surge capacity as possible.
4. Do use a tangential entry for pneumatic conveying of fluidizable solids whenever possible, or provide a deflector plate.
5. Don't split material into various bins from a belt conveyor, since the material may be segregated on the belt.
6. Don't design a nonsymmetric multiple-outlet bin.
7. Don't use free-fall chutes to transfer materials having differing friction angles unless there is a mixing device downstream.
8. Don't charge a mixture of fine fluidizable powder and nonfluidizable coarse particles from a pneumatic conveying line with a vertical downspout.

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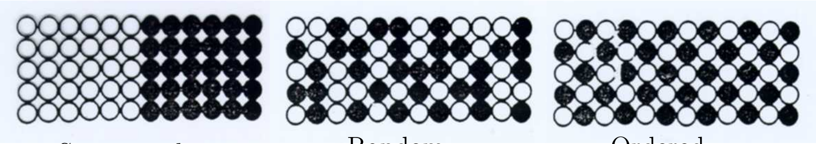
What is a mixture?

Type of mixtures



Segregated Random Ordered

What is a mixture?

The true composition of a mixture p is often not know, but by sampling N times, each with value y_i, we can obtain an estimate, $\bar{y}$

$$\text{mean} : \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

$$\text{standard deviation} : \sigma = \frac{\sum_{i=1}^{N}(y_i - \bar{y})}{N-1}$$

$$\text{random case } : \sigma_r^2 = \frac{p(1-p)}{n}$$

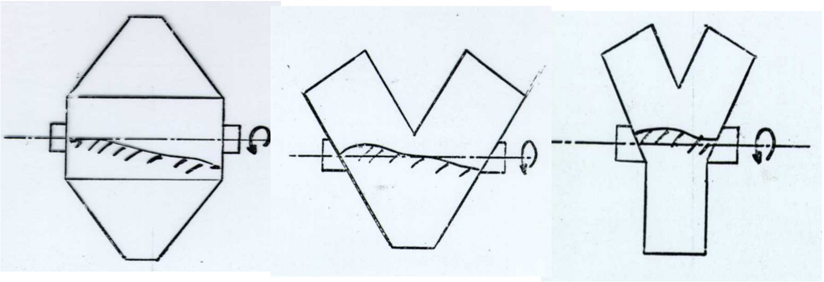
where n is the number of particles in the samples.

$$\text{segregated case } : \sigma_o^2 = p(1-p)$$

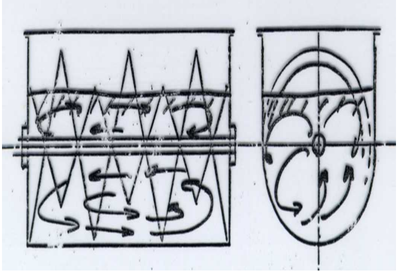
Mixing indices

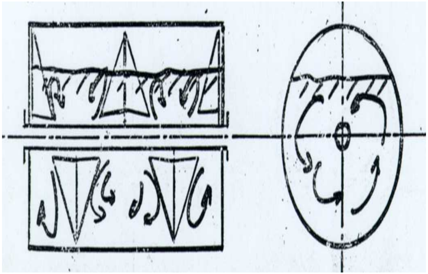
- Lacey mixing index $M_L = \frac{\sigma_0^2 - \sigma^2}{\sigma_0^2 - \sigma_r^2}$
- Problem with M_L is practical values only lie in range $0.75 - 1.0$
- Poole, Taylor & Wall mixing index $M_P = \frac{\sigma_r}{\sigma}$
- This gives better discrimination
- Many, many other indexes exist
- Note σ measured by sampling may not be the true mixture σ. This brings us to the topic of confidence intervals which will not be discussed here.

Tumbling mixer

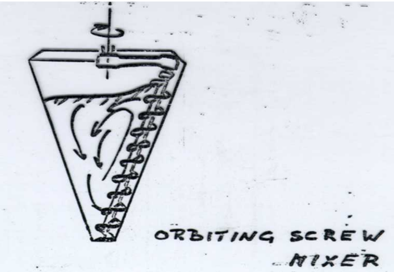


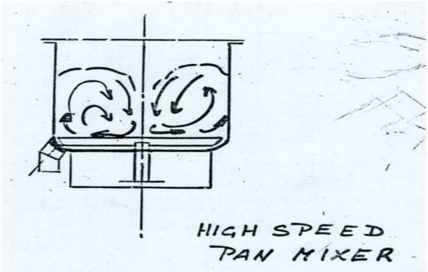
Ribbon blade mixers



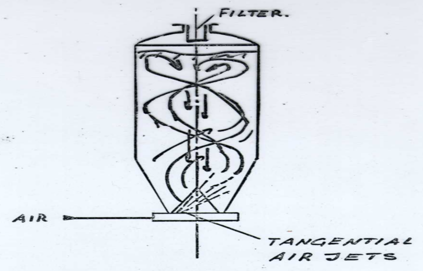


Rotating mixers

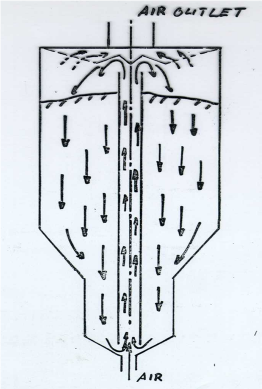




Air-jet mixer



Pneumatic mixer



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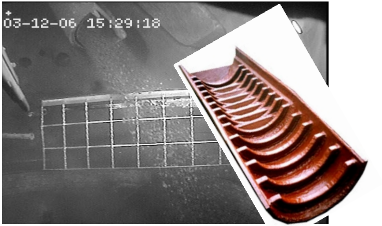
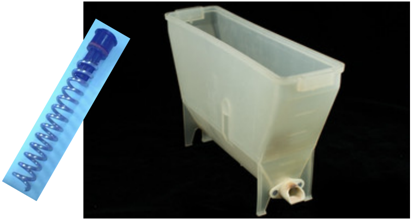
Grains in industry


Heat-drying in tumbler


Silo flow

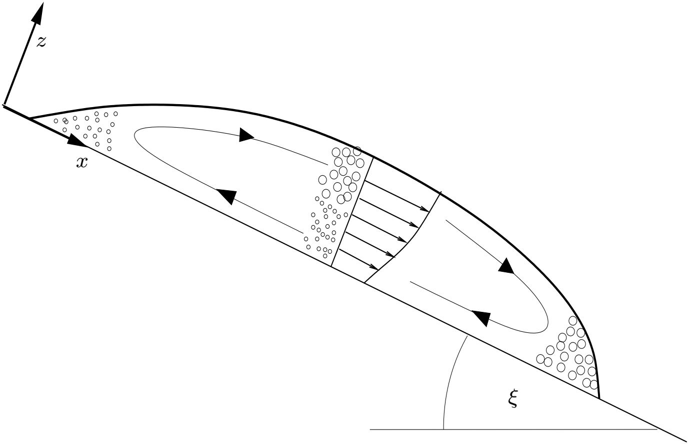


Rotating blast-furnace chute


Vending machine canister

Motivation

Co-ordinate setup



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Basic concepts

- Follow structure of Savage & Lun (1988)
- Two constituents mixture theory
  - Small particles, s
  - Large particles, b
- With volume fractions

$$0 \leq \phi^{\mu} \leq 1, \quad \mu = s, b$$

and

$$\phi^{s} + \phi^{b} = 1$$

(Gray & Thornton, 2005, Proc. Royal Soc.)

Mixture theory - basic postulate

The basic mixture postulate

States that every point in the mixture is ‘occupied simultaneously by all constituents’

- Mixture theory deals with **partial** variables defined per unit mixture volume.
- Whereas **intrinsic** variables are defined by unit constituent volume.
- So each constituent we can define a local volume fraction ϕ^ν and clearly

$$\sum_{\nu=1}^{n} \phi^\nu = 1$$

- Hence the sum across all constituents of an intrinsic variables is equal to the bulk quaintly i.e. density

Mixture theory

- Mass balance

$$\frac{\partial \rho^\nu}{\partial t} + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu) = 0,$$

- Momentum balance

$$\rho^\nu \frac{D\boldsymbol{u}^\nu}{Dt} = -\nabla p^\nu + \rho^\nu \boldsymbol{g} + \boldsymbol{\beta}^\nu.$$

where
$\rho^\nu \boldsymbol{g}$ is the gravitational acceleration
$\boldsymbol{\beta^\nu}$ is the interaction drag
ρ^ν, p^ν and $\boldsymbol{u}^\nu$ are partial variables defined per unit **mixture** volume

Mixture theory - key relations

- The internal drags must sum to zero

$$\Sigma_\nu \beta^\nu = 0$$

- The partial and intrinsic density are related by simple linear volume fraction scaling

$$\rho^\nu = \phi^\nu \rho^{\nu*}$$

- The partial and intrinsic velocities are the same

$$\boldsymbol{u}^\nu = \boldsymbol{u}^{\nu*}$$

- The pressures are related by an unknown function normally taken to be the volume fraction

$$p^\nu = \phi^\nu p^{\nu*} \quad f^\nu(\phi^\nu) = p^{\nu*}/p$$

where * denotes an intrinsic variable.

Assumptions

- Bulk flow incompressible
- Normal acceleration terms are negligible
- Interaction drag is Darcy type, β
- Kinetic sieving process
    - Modelled by a non-linear (partial) pressure, $f(\phi)$
    - Different forms suggested

(Gray & Thornton, 2005, Proc. Royal Soc.)

Stress scalings

Gray & Thornton	:	$f^\nu = 1 - B(1-\phi^\nu)$
Marks, Rognon & Einav	:	$f^\nu = \frac{s^\nu}{\sum s^\nu \phi^\nu}$
Tunuguntla, *et al.*	:	$f^\nu = \frac{(s^\nu)^3}{\sum (s^\nu)^3 \phi^\nu}$
Gajjar & Gray	:	$f^\nu = A_C(1-\phi^\nu)(1-C\phi^\nu)$

where s is the size ratio.

$$f^\nu(\phi^\nu) = \sigma^{\nu*}/\sigma \quad F = (f^\nu - \phi^\nu)/c\frac{\partial \sigma_{zz}}{\partial z}$$

(Gray & Thornton, 2005, JFM)
(Marks, Rognon & Einav, 2012, JFM)
(Tunuguntla, Bokhove & Thornton, 2014, JFM)
(Gajjar & Gray, 2014, JFM)

The binary segregation equation

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi u)+\frac{\partial}{\partial y}(\phi v)+\frac{\partial}{\partial z}(\phi w)-S_r\frac{\partial}{\partial z}(F[\phi])=\frac{\partial}{\partial z}\left(D_r\frac{\partial \phi}{\partial z}\right)$$

where ϕ : is the volume fraction of small particles
u, v, w : down slope/cross slope/normal velocity components
S_r : is a dimensionless segregation rate
and D_r : is a dimensionless diffusion rate.

G & T : $F[\phi]=(\phi(1-\phi))$

T, B & T : $F[\phi]=\dot{\gamma}(\hat{s}^a-\hat{\rho})\left[\frac{\phi(1-\phi)}{\phi+(1-\phi)\hat{s}^a}\right]$ $\hat{s}=\frac{s^2}{s^1}, \hat{\rho}=\frac{\rho^{2*}}{\rho^{1*}}$.

Note : Experiments and simulations show $D_r/S_r \approx 1/20$.

(Gray & Thornton, 2005, Proc. Royal Soc.)
(Tunuguntla *et al.*, 2014, JFM)

The binary (dimensionless) segregation equation

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi u)+\frac{\partial}{\partial y}(\phi v)+\frac{\partial}{\partial z}(\phi w)-S_r\frac{\partial}{\partial z}(F[\phi])=\frac{\partial}{\partial z}\left(D_r\frac{\partial \phi}{\partial z}\right)$$

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$F(\phi)$ is the segregation flux function.

Note : Experiments and simulations show
$1/P_e = D_r/S_r \approx 1/20$.

(Gray & Thornton, 2005, Proc. Royal Soc.)
(Tunuguntla, Weinhart & Thornton, 2017, CPM)

The binary segregation equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial y}(\phi v) + \frac{\partial}{\partial z}(\phi w) - S_r \frac{\partial}{\partial z}(F[\phi]) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi}{\partial z}\right)$$

- Bridgwater suggested form: $F = \phi g(\phi)$
- Savage & Lun derived a very complex form for F
- Dolgunin & Ukolov suggested form: $F = B\phi(1-\phi)$
- Mixture theory framework proposed: $F = B\phi(1-\phi)\rho g$
- Driven by kinetic stress: $F = -B(\phi(1-\phi))\frac{1}{\rho}\frac{\partial \sigma^k_{zz}}{\partial x}$
- Density segregation: $F = \frac{g}{c}\dot{\gamma}(\hat{s} - \hat{\rho})\left[\frac{\phi(1-\phi)}{\phi+(1-\phi)\hat{s}}\right]$
- Measured from particle simulations:
  - $F = \frac{g}{c}\dot{\gamma}\left(\hat{s}^3 - \hat{\rho}\right)\left[\frac{\phi(1-\phi)}{\phi+(1-\phi)\hat{s}^3}\right]$
  - $F = Bd_s \ln\left(\frac{d_l}{d_s}\right)\dot{\gamma}\phi(1-\phi)$

The binary segregation equation

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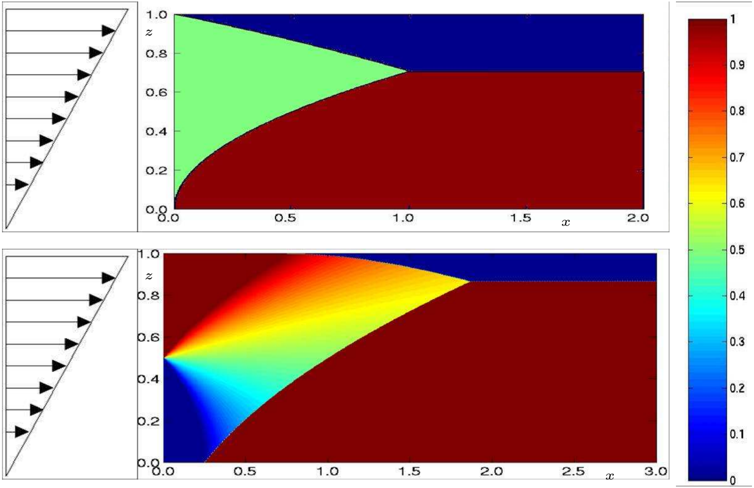
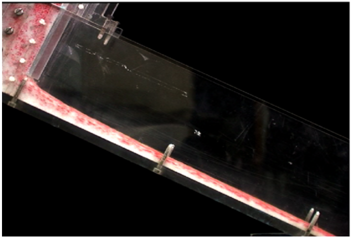
(Bridgwater, Foo & Stephens, 1985, Powder Tech.)
(Bridgwater *et al.*, 1985; Savage & Lun, 1988, Powder Tech./JFM)
(Bridgwater *et al.*, 1985; Dolgunin & Ukolov, 1995; Savage & Lun, 1988, Powder Tech./JFM)
(Gray & Thornton, 2005; Gray & Chugunov, 2006, Proc. Royal Soc./JFM)
(Fan & Hill, 2011, NJP)
(Marks *et al.*, 2012, JFM)
(Tunuguntla *et al.*, 2014; Schlick, Isner, Freireich, Fan, Umbanhowar, Ottino & Lueptow, 2016, Comp. Part. Mech./JFM)

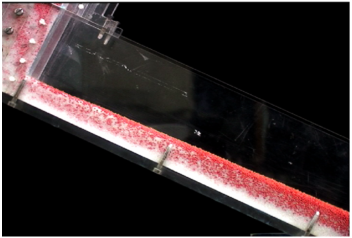
Exact Solutions: GT style flux

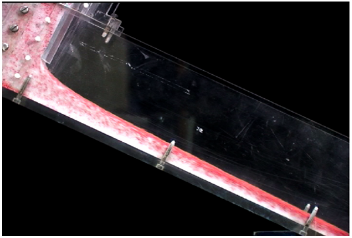


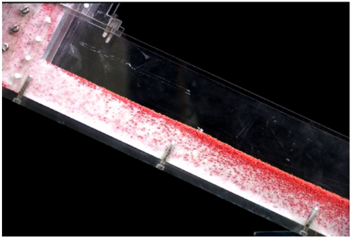
(Gray & Thornton, 2005; Thornton, Gray & Hogg, 2006, Proc.Roy.Soc./JFM)

Experimental comparison









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(Tunuguntla *et al.*, 2017, CPM)

The force model

- Discrete particle model governed by Newtonian mechanics:

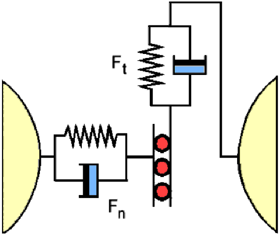
$$m_i \frac{\mathrm{d}^2 \vec{x}_i}{\mathrm{d}t^2} = \vec{f}_i$$

- Contact forces and body forces:

$$\vec{f}_i = \sum_j \vec{f}_{ij} + \vec{b}_i,$$

- Contact force model:

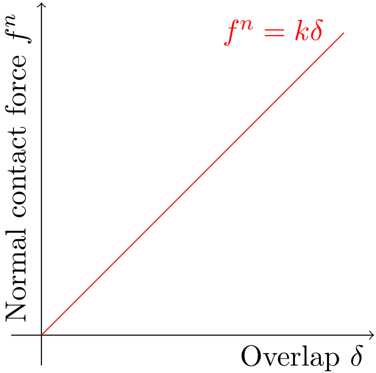
$$\vec{f}_{ij} = f_{ij}^n \vec{n} + f_{ij}^t \vec{t},$$



$$f_{ij}^n = k\delta_{ij} + \gamma v_{ij}^n, \quad f_{ij}^t = -min(\mu f_{ij}^n, k^t \delta_{ij}^t + \gamma^t v_{ij}^t)$$

(Luding, 2008, Enviro. and Civil. Eng.)

Linear(-ised) elastic normal force



- Appropriate for larger particles, upscaled systems.
- Simple to analyse, efficient.
- ! Stiffness has to be sufficiently high ($\delta < \frac{R}{100}$).
- Constant t_c, restitution ϵ if diss. coeff. $\gamma = const.$

Contact Properties

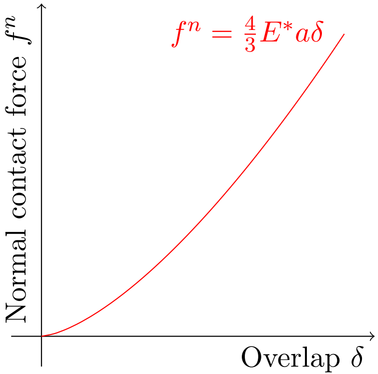
Can relate these properties to a restitution coefficient r and contact time t_c

$$r = e^{\frac{-\pi\gamma}{\sqrt{4km_{ij}-\gamma^2}}}$$

$$t_c = \frac{2m_{ij}\pi}{\sqrt{4km_{ij}-\gamma^2}}$$

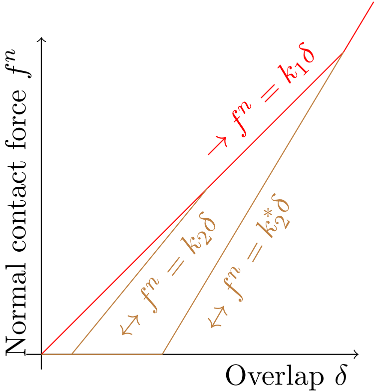
We define γ and k for each pair of particle-interactions such that r and t_c are the same.

(Luding, 2008, Enviro. and Civil. Eng.)



- Stiffness depends on modulus $E^* = [\frac{(1-\nu_1^2)}{E_1} + \frac{(1-\nu_2^2)}{E_2}]^{-1}$ and contact radius $a \approx \sqrt{R\delta}$.
- Appropriate for small particles ($< 100\mu$m).
- Variable collision time.
- Yields constant restitution coefficient for $\gamma \propto \sqrt{Eam}$.

Linearised elastoplastic normal force



- Appropriate for larger, plastic deformation.
- k_2 increasing towards maximum k_2^* (plastic yield).

Introduction to MercuryDPM

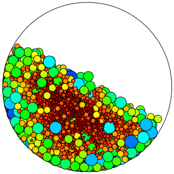
The simulations presented are done in MercuryDPM, our open-source code. Features :

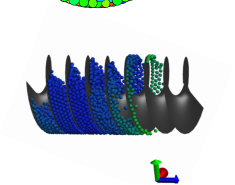
- Hierarchical Grid contact detection algorithm
- Built-in coarse-graining statistical package
- Access to continuum fields in real time
- Contact laws for granular materials
- Simple C++ implementation
- Complex walls

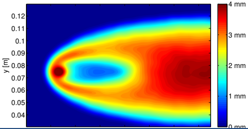
Currently available as a beta version from `http://MercuryDPM.org`

Particle simulations with MercuryDPM

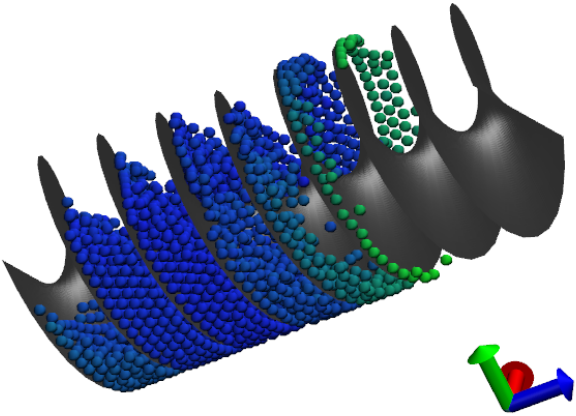
- **Fast**
  Contact detection algorithm
  allows polydisperse simulations
- **Flexible**
  Support for complex walls
  and boundary conditions
- **Accurate**
  Coarse-graining technique
  to evaluate continuum fields
- **Open-source**
  Available at MercuryDPM.org







Screw Feeder and Mixers



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Coarse-Graining

- Microscopic density field,

$$\rho_{\text{micro}}(\vec{r}) = \sum_{i=1}^{n} m_i \delta_i(\vec{r} - \vec{r}_i),$$

- Macroscopic density field,

$$\rho(\vec{r}) = \sum_{i=1}^{n} m_i \phi(\vec{r} - \vec{r}_i),$$

with coarse-graining function ϕ, *e.g.*

$$\phi_\omega^G(\vec{r}) = \frac{1}{(\sqrt{2\pi}\omega)^d} \exp\left(-\frac{|\vec{r}|^2}{2\omega}\right).$$

- Other fields defined to be consistent with macroscopic equations

---

(Goldhirsch, 2010, Granular Matter)

Mass Conservation

- mass conservation, $\frac{\partial}{\partial t}\rho + \nabla(\rho\vec{V}) = 0$,
- define the velocity field as

$$\vec{V} = \frac{\vec{j}}{\rho}, \text{ where } \vec{j} = \sum_{i=1}^{n} m_i \vec{v}_i \phi(\vec{r} - \vec{r}_i).$$

- This is compatible with the macroscopic field

(Goldhirsch, 2010, Granular Matter)

Momentum balance

- momentum equations, $\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{V}\cdot\nabla(\rho\vec{V}) = \rho\vec{g} + \nabla\vec{\sigma}$,
- define the stress tensor by $\vec{\sigma} = \vec{\sigma}^c + \vec{\sigma}^k$ with contact and kinetic stress,

$$\sigma^c_{\alpha\beta} = \frac{1}{2}\sum_{\mathrm{c}} f_{ij\alpha} r_{ij\beta} \int_0^1 \phi(\vec{r} - (\vec{r}_i + s\vec{r}_{ij}))\,ds,$$

$$\sigma^k_{\alpha\beta} = \sum\nolimits_{i=1}^{n} m_i v'_{i\alpha} v'_{i\beta} \phi(\vec{r} - \vec{r}_i),\ \ \alpha,\beta = 1,2,3,$$

with fluctuation velocity $\vec{v}_i' = \vec{v}_i - \vec{V}$.

- Definitions are compatible with the momentum balance.
- Can be done for other fields (boundary forces, drag, partial fields for multiphase flows).

(Goldhirsch, 2010, Granular Matter)

CG: Multi-components

- Stresses are determined s.t. momentum is conserved

$$\partial_t(\rho^\nu \vec{v}^\nu) + \nabla \cdot (\rho^\nu \vec{V}^\nu \vec{V}^\nu) = \nabla \cdot \boldsymbol{\sigma}^\nu + \vec{b} + \vec{t} + \vec{\beta}^\nu.$$

- Total partial stress, $\sigma^\nu = \sigma^{k,\nu} + \sigma^{c,\nu}$ where

$$\boldsymbol{\sigma}^{k,\nu} = \sum_{i \in \mathcal{F}^\nu} m_i \vec{v}_i{}' \vec{v}_i{}' \phi_i \text{ with } \nu = 1, 2 \;\; \&$$

$$\boldsymbol{\sigma}^{c,\nu} = \sum_{i \in \mathcal{F}^\nu} \sum_{\substack{j \in \mathcal{F}^\nu \\ j \neq i}} \vec{f}_{ij}\, \vec{a}_{ij} \psi_{ij} + \sum_{i \in \mathcal{F}^\nu} \sum_{j \in \mathcal{F}/\mathcal{F}^\nu} \vec{f}_{ij}\, \vec{a}_{ij} \psi_{ij} + \sum_{i \in \mathcal{F}^\nu} \sum_{j \in \mathcal{W}} \vec{f}_{ij}\, \vec{a}_{ij} \psi_{ij},$$

with $\psi_{ij} = \int_0^1 \phi(\vec{r} - \vec{r}_i + s\vec{a}_{ij}) ds$ and $\vec{a}_{ij}$ is the branch vector

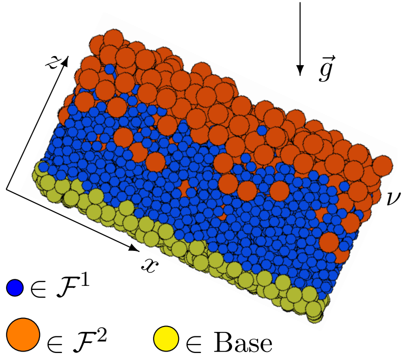
(Tunuguntla, Thornton & Weinhart, 2016, CPM)

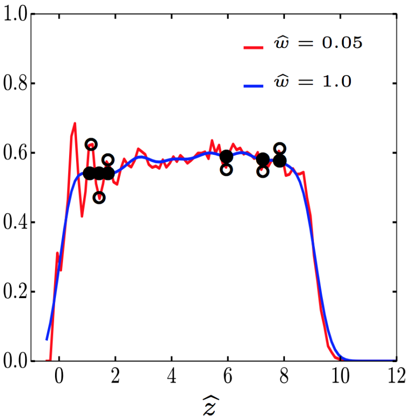
What is macroscopic field?

- We have a smoothing length w, how do we choose it?
- If w too small you see individual particles - Not the macroscopic field.
- If w too large you average over macroscopic variations in the field.
- Between these two values there should be a plateau, this is the macroscopic field.

(Goldhirsch, 2010, Granular Matter)

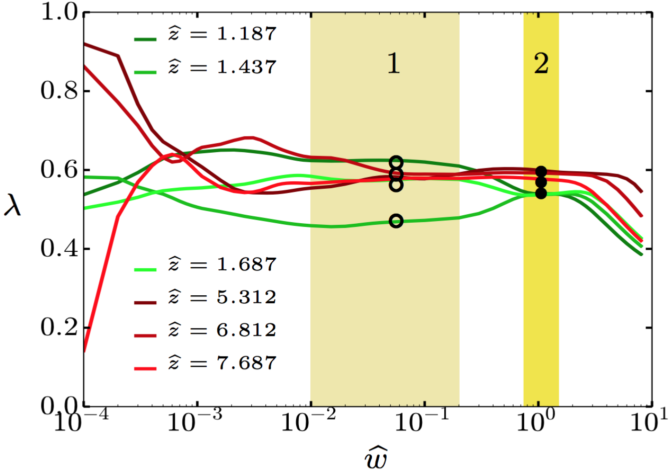
CG Application: Steady bidisperse flows





(Tunuguntla *et al.*, 2016, CPM)

Scale independence



(Tunuguntla *et al.*, 2016, CPM)

Advantages of MercuryCG

1. Already freely available in open-source code MercuryDPM.
2. Local mass/momentum balance is satisfied exactly, (for any smoothing width w, no ensemble averaging required).
3. Gives continuum field everywhere; no grid.
4. Only one parameter to determine.
5. Can account for boundary interactions.
6. Extended to granular mixtures.

(Thornton & Weinhart, 2009-2018, http://MercuryDPM.org)
(Goldhirsch, 2010, Gran. Mat. 2010)
(Weinhart, Hartkamp, Thornton & Luding, 2013*a*, Phys of Fluids 2013)
(Weinhart, Thornton, Luding & Bokhove, 2012, Granular Matter 2013)
(Tunuguntla *et al.*, 2017, Comp. Part. Mech. 2016)

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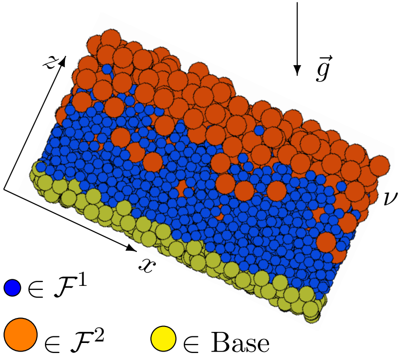
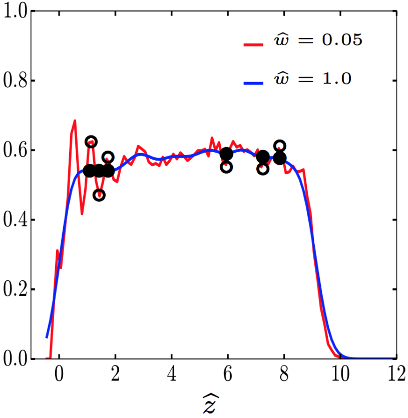
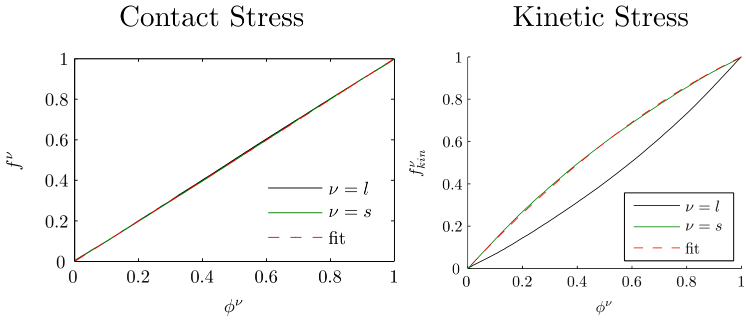
CG Application: Steady bidisperse flows





(Tunuguntla *et al.*, 2016, CPM)

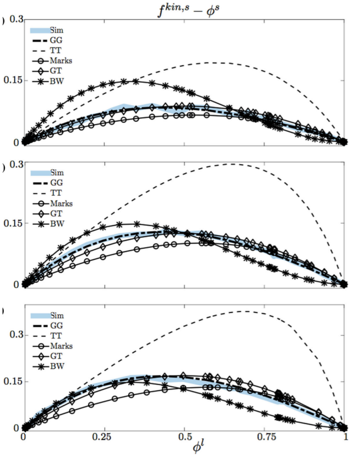
Stress distribution



- Variation in kinetic stress stronger.
- However both stresses show correct dependence.
- We will assume segregation driven by kinetic stress.

(Weinhart, Luding & Thornton, 2013b, P&G 2013)
(Fan & Hill, 2011, NJP 2011)

Time dependent CG kinetic stress



(Tunuguntla *et al.*, 2017, CPM)

- Comparison shown for three size-ratios 1.3, 1.5, 1.7
- Best fit is model of Gajjar & Gray.
- Should be noted free parameter A_γ is size dependent.

The binary segregation equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial y}(\phi v) + \frac{\partial}{\partial z}(\phi w) - S_r \frac{\partial}{\partial z}(F[\phi]) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi}{\partial z}\right)$$

where ϕ : is the volume fraction of small particles
u, v, w : down slope/cross slope/normal velocity components
S_r : is a dimensionless segregation rate
and D_r : is a dimensionless diffusion rate.

$$F[\phi] = -A\phi(1-\phi)(1-\kappa[s]\phi)\frac{1}{\rho}\frac{\partial \sigma^k_{zz}}{\partial x}$$

Note : In chute flows $D_r/S_r \approx 1/20$ and $\frac{\partial \sigma^k_{zz}}{\partial x} \approx C\rho g$

(Gray & Thornton, 2005, Proc. Royal Soc.)
(Gray & Thornton, 2005; Gray & Chugunov, 2006, Proc. Royal Soc./JFM)
(Fan & Hill, 2011, NJP)
(Fan & Hill, 2011; Gajjar & Gray, 2014, JFM/NJP)
(Tunuguntla *et al.*, 2017, Comp. Part. Mech.)

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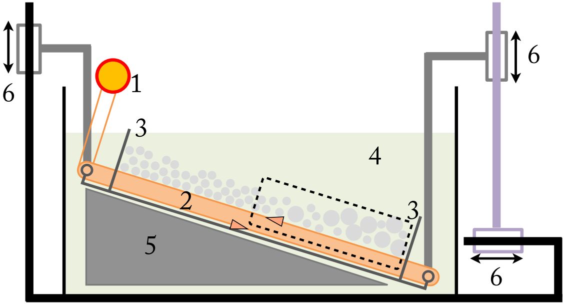
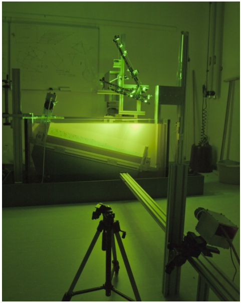
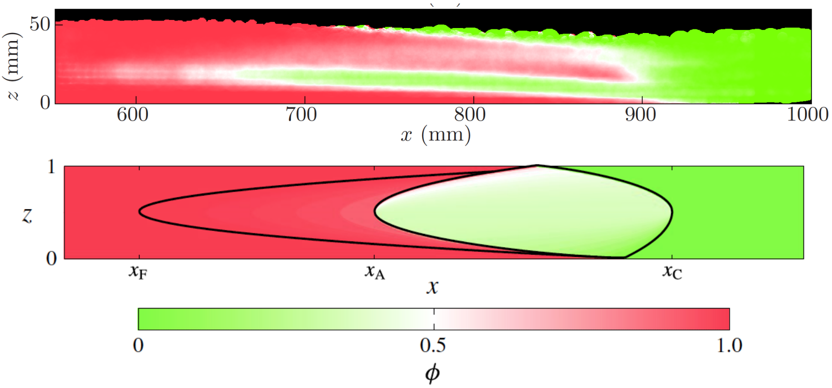
3D Moving Channel and RIMS





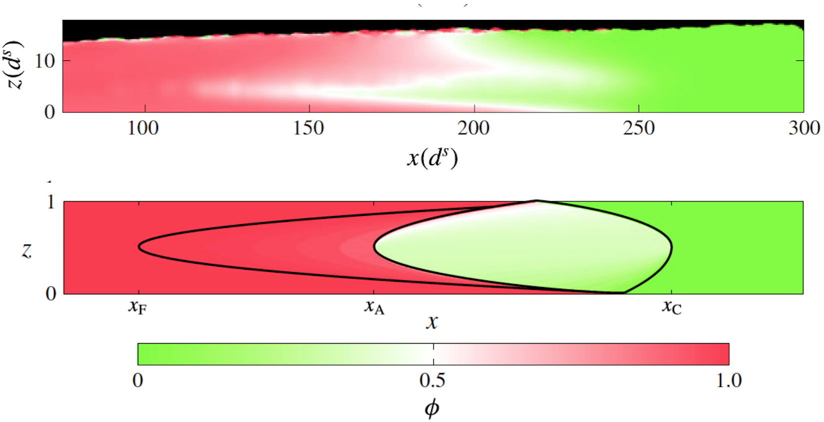
(van der Vaart, Thornton, Johnson, Weinhart, Jing, Gajjar, Gray & Ancey, 2018, Gran. Mat.)

RIMS Experiments



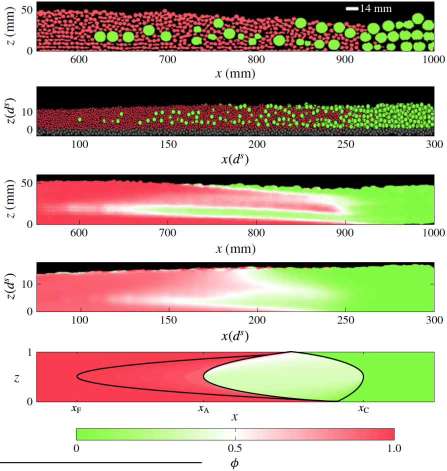
(van der Vaart *et al.*, 2018, Gran. Mat.)

Particle Simulations



(van der Vaart *et al.*, 2018, Gran. Mat.)

Breaking Wave Comparison



(van der Vaart *et al.*, 2018, Gran. Mat.)
(Gajjar, van der Vaart, Thornton, Johnson, Ancey & Gray, 2016, JFM)

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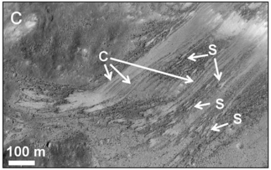
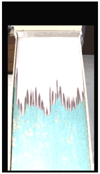
Segregation patterns in Geomechanics

Scree slope
Lake District, U.K.

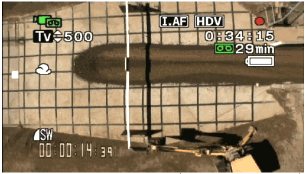
Scree slope
Nordaustlandet (Norway)



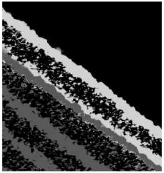
Debris flow
The Moon



Fingering



Elongated run-off (Bulbous head)

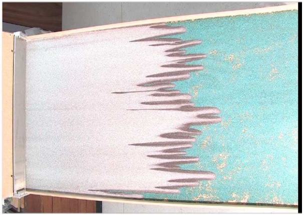


Stratification

(Kokelaar, Bashi, Joy, Viroulet & Gray, 2018, JGR Planets)
(Johnson, Kokelaar, Iverson, Logan, Lahusen & Gray, 2012, JGR)
(Gray & Ancey, 2009, JFM)

Mount Ruapehu avalanche

Experimental results



Shallow water like theories

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0,$$

$$\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}\left(h\bar{u}^2\right) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) + \frac{1}{2}\frac{\partial}{\partial x}\left(gh^2\cos\theta\right) = gh\left(\sin\theta - \mu\frac{\bar{u}}{\sqrt{\bar{u}^2+\bar{v}^2}}\cos\theta\right)$$

$$\frac{\partial}{\partial t}(h\bar{v}) + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(h\bar{v}^2\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(gh^2\cos\theta\right) = gh\left(-\mu\frac{\bar{v}}{\sqrt{\bar{u}^2+\bar{v}^2}}\cos\theta\right)$$

(Savage & Hutter, 1989, JFM)

The Pouliquen friction law

- The law

$$\mu^\nu(h, \boldsymbol{u}) = \tan\delta_1^\nu + \frac{\tan\delta_2^\nu - \tan\delta_1^\nu}{\frac{\beta^\nu h}{A^\nu d^\nu F} + 1}$$

- δ_1^ν is minimum angle for the material to flow
- δ_2^ν is the maximum angle at which steady uniform flows can be observed
- A^ν is a characteristic length scale
- $\beta^\nu = 0.136$ (for most materials)
- d^ν is the diameter of the particles
- F is the Froude number

(Pouliquen, 1999, Phys. Fluids 11 (3))
(Denissen, Weinhart, te Voortwis, Luding, Gray & Thornton, 2019, JFM)

The binary segregation equation

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi u)+\frac{\partial}{\partial y}(\phi v)+\frac{\partial}{\partial z}(\phi w)-S_r\frac{\partial}{\partial z}(F[\phi])=\frac{\partial}{\partial z}\left(D_r\frac{\partial \phi}{\partial z}\right)$$

where ϕ : is the volume fraction of small particles
u, v, w : down slope/cross slope/normal velocity components
S_r : is a dimensionless segregation rate
and D_r : is a dimensionless diffusion rate.

$$F[\phi]=-A\phi(1-\phi)(1-\kappa[s]\phi)\frac{1}{\rho}\frac{\partial \sigma^k_{zz}}{\partial x}$$

Note : In chute flows $D_r/S_r \approx 1/20$ and $\frac{\partial \sigma^k_{zz}}{\partial x} \approx C\rho g$

(Gray & Thornton, 2005, Proc. Royal Soc.)
(Gray & Thornton, 2005; Gray & Chugunov, 2006, Proc. Royal Soc./JFM)
(Fan & Hill, 2011, NJP)
(Fan & Hill, 2011; Gajjar & Gray, 2014, JFM/NJP)
(Tunuguntla *et al.*, 2017, Comp. Part. Mech.)

Adding in segregation dynamics

- Depth integrate the 3D segregation equation
- Introduce the averages

$$\bar{\phi} = \frac{1}{h}\int_0^1 \phi \, \mathrm{d}z$$

- Assume segregation is instantaneous i.e. take the limit $S_r \to \infty$ and that the velocity profile is $u = \bar{u}\left(\alpha + 2(1-\alpha)\left(\frac{z-b}{h}\right)\right)$.
- Leads to

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\bar{u}\bar{\phi}) + \frac{\partial}{\partial y}(h\bar{v}\bar{\phi}) =$$
$$(1-\alpha)\left(\frac{\partial}{\partial x}(h\bar{u}(\bar{\phi}-\bar{\phi}^2)) + \frac{\partial}{\partial y}(h\,\bar{v}(\bar{\phi}-\bar{\phi}^2))\right) = 0.$$

The fully coupled 2D system

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0,$$

$$\frac{\partial}{\partial t}(\alpha hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\cos\theta\right) = gh\left(\sin\theta - \mu\frac{u}{\sqrt{u^2+v^2}}\cos\theta\right)$$

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}\left((hu\bar{\phi}) + (1-\alpha_s)(hu(\bar{\phi}-\bar{\phi}^2))\right)$$

with

$$\mu = \bar{\phi}\mu^s + (1-\bar{\phi})\mu^l \quad \alpha = \frac{1}{3}(1-\alpha_s)^2 + 1$$

and

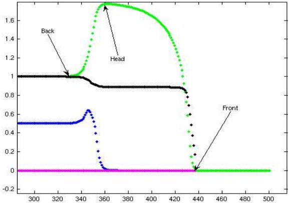
$$\mu^\nu(h, \boldsymbol{u}) = \tan\delta_1^\nu + \frac{\tan\delta_2^\nu - \tan\delta_1^\nu}{\frac{\beta^\nu h}{A^\nu d^\nu F} + 1}$$

(Denissen *et al.*, 2019, JFM)

The bulbous head solution

ϵ	0.1	ϕ_{inflow}	0.9
δ_1^s	$20°$	δ_2^s	$30°$
δ_1^l	$27°$	δ_2^l	$37°$
α	0.0	$L_l = L_s$	1.0
x-length	500	y-length	20
no. points x	500	no. points y	500

The bulbous head solution



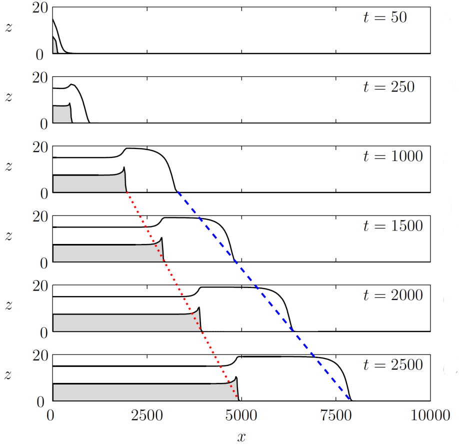
- By considering mass balance we can show

$$U_{front} = U_{inflow} \left(1 - \alpha\phi_0 + \phi_0^2 - \alpha\phi_0^2\right)$$

$$U_{back} = U_{inflow} \left(\alpha + (1-\alpha)\,\phi_0\right)$$

- Since the front consists of a pure phase of large particles its shape is given by Pouliquen's finger solutions. Hence

The bulbous head solution



(Denissen *et al.*, 2019, JFM)

One-dimensional travelling wave solution

We will seek one-dimenstional travelling solution. Hence making the transformation

$$\hat{x} = x - u_f t, \quad \frac{\partial}{\partial y} = 0. \quad \hat{t} = t$$

It can be shown that the equation for $\bar{u}$ can be reduced to the following o.d.e.

$$\frac{d\bar{u}}{d\hat{x}} = \frac{s}{\left((1 - \bar{u}_f) - \epsilon \cos\theta \frac{(1-u_f)}{(\bar{u}-u_f)^2}\right)},$$

where

$$s = \mu \frac{u}{\sqrt{u^2 + v^2}} \cos\theta$$

Relationships with $\bar{u}$ and h

Once you have solved the o.d.e for $\bar{u}$, both h and C are similar given by the following algebraic equations

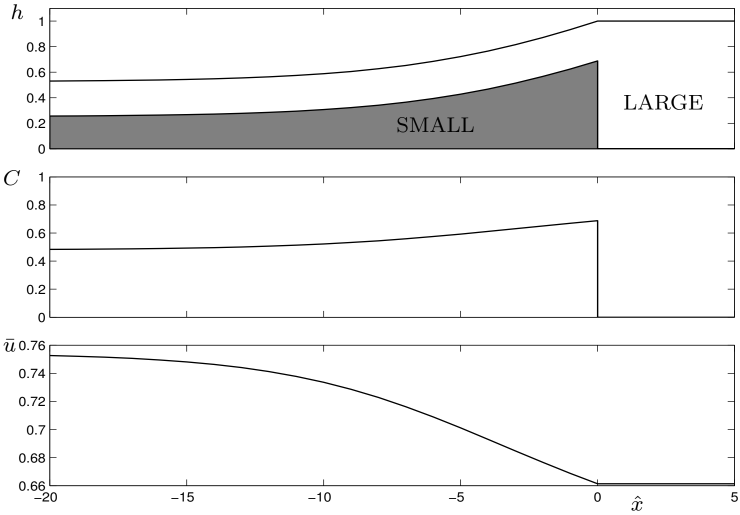
$$h = \frac{1-u_f}{\bar{u}-u_f}.$$

$$C^2 + c_1 C + c_0 = 0$$

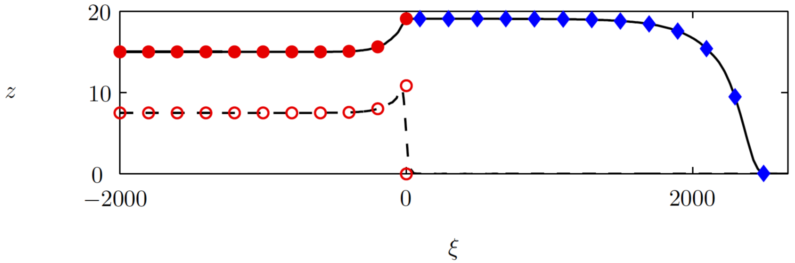
where

$$c_1 = \frac{\alpha\bar{u}-u_f}{(1-\alpha)\,\bar{u}} \quad \text{and} \quad c_0 = \frac{\bar{u}-u_f}{\bar{u}}\left(\frac{C_0\,(1-C_0)}{1-u_f} - \frac{C_0}{1-\alpha}\right).$$

Solution


h
1
0.8
0.6
0.4
0.2
0
LARGE
SMALL
C
1
0.8
0.6
0.4
0.2
0
ū
0.76
0.74
0.72
0.7
0.68
0.66
−20
−15
−10
−5
0
5
x̂

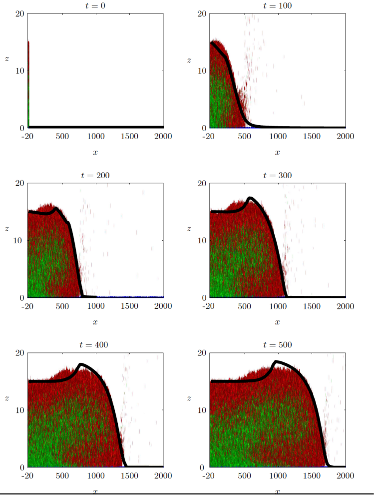
Travelling wave solution versus DGFEM



- Black line is DGFEM solution
- Blue line is a Saingier (Pouliquen) finger solution
- Red line is a segregation travelling wave solution

(Denissen *et al.*, 2019, JFM)
(Saingier, Deboeuf & Lagree, 2016, Phys. Fluids)

DGFEM versus particle simulations

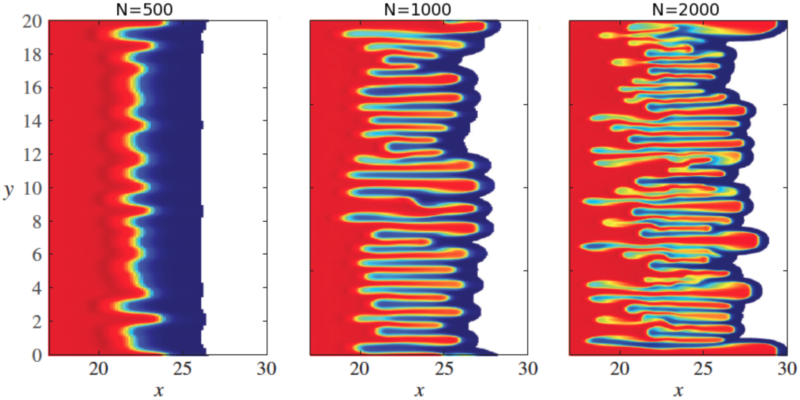


(Denissen *et al.*, 2019, JFM)

- Black line is DGFEM solution
- Red/green are large/small particles from a simulation
- There are no fitting parameters
- Parameters of model are 'measured'
- This is very compressed the flow is very long and thin

Grid dependence

So problem solved, well no.



(Woodhouse *et al.*, 2012, JFM)

Results

Asymptotic results for high k_x show

- That for $\bar{u_0} \neq u_s$ to leading order eigenvalues are purely imaginary for $k_x >> 1$.
- However, on the curve $\bar{u_0} = u_s \quad \sigma \approx k^{1/2}$ for $k_x >> 1$.
- Linear stability analysis of a constant solution shows system is ill posed on a single curve.
- Both fingering and propagating head solutions can be formed
- The number of fingers produced is grid **dependent**
- However, it is linear unstable at high wave numbers
- Shallow layer of fluid on an incline has a similar problem
- System can be stabilised by adding viscous to the momentum balance

(Woodhouse *et al.*, 2012, JFM)

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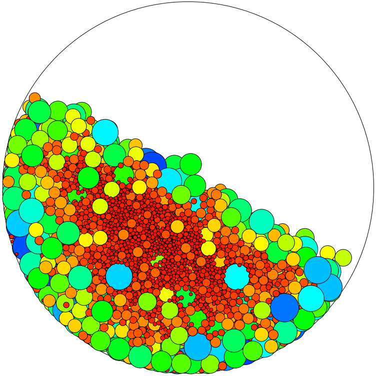
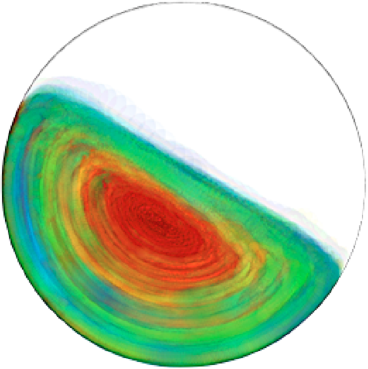
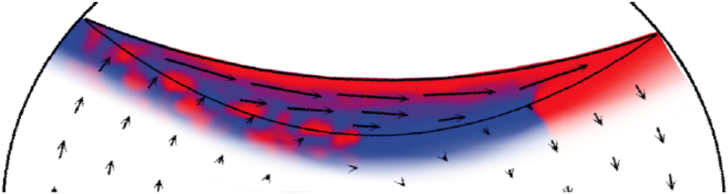
DPM of segregation in a rotating drum



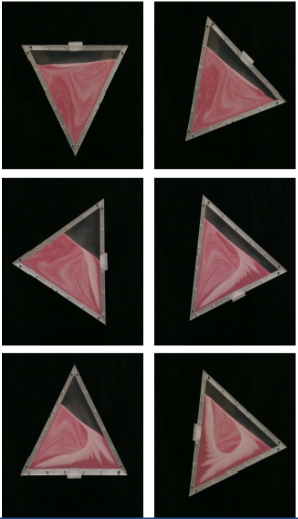


Schematic of segregating in rotating drum

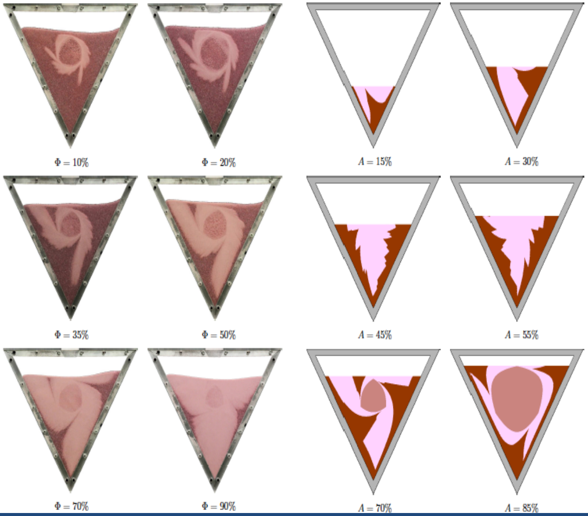


Large particles in red
Small particles in blue

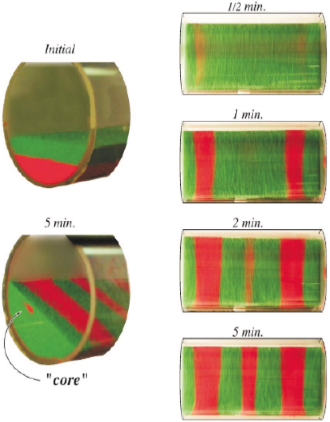
Segregating in a Rotating Triangle

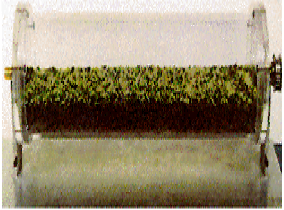


Final Patterns in Rotating Triangle



Segregating in Rotating Cylinder





Initial Mixture of Uncooked Rice and Split Peas

After Rotation About Horizontal Axis at 15 rpm for 2 hours

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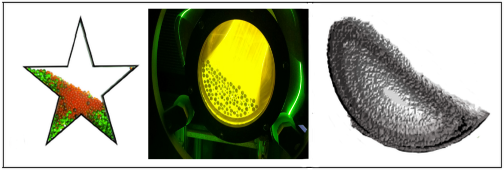

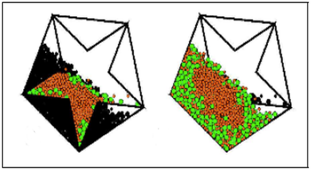

- Discussed definition of mixed state
- Showing different industrial mixers
- Showed a family of models for granular segregation
- Showed how to use DPM to calibrate and validate such models
- Coupled segregation and bulk flow models
- Showed how a reduced version of this model can be applied to rotating drums
- Consider axial patterns in long rotating cylinders
- Coupled the segregation model with shallow water equations to consider geophysical problems

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