Applied Numerical Mathematics 76 (2014) 101–131

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Asymptotically exact discontinuous Galerkin error estimates for linear symmetric hyperbolic systems

Slimane Adjerid [a,*], Thomas Weinhart [b]

[a] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, United States
[b] Multiscale Mechanics, CTW and MESA+, University of Twente, PO Box 217, 7500AE Enschede, The Netherlands

ARTICLE INFO

Article history:
Received 6 March 2011
Received in revised form 2 June 2013
Accepted 28 June 2013
Available online 6 August 2013

Keywords:
Discontinuous Galerkin method
Symmetric hyperbolic systems
A posteriori error estimation
Superconvergence

ABSTRACT

We present an *a posteriori* error analysis for the discontinuous Galerkin discretization error of first-order linear symmetric hyperbolic systems of partial differential equations with smooth solutions. We perform a local error analysis by writing the local error as a series and showing that its leading term can be expressed as a linear combination of Legendre polynomials of degree p and $p+1$. We apply these asymptotic results to observe that projections of the error are pointwise $\mathcal{O}(h^{p+2})$-superconvergent in some cases. Then we solve relatively small local problems to compute efficient and asymptotically exact estimates of the finite element error. We present computational results for several linear hyperbolic systems in acoustics and electromagnetism.

1. Introduction

First-order hyperbolic systems arise in many areas of continuum physics when fundamental balance laws are formulated (such as the conservation of mass, momentum, or energy) and if other small-scale, dissipative mechanisms can be neglected. Many of these systems can be written in symmetric form, such as Maxwell's equations of electromagnetism, the wave equation, and the two-dimensional Euler's equation modeling gas dynamics.

The discontinuous Galerkin (DG) finite element method was first used to solve the neutron equation [28] and then studied for initial-value problems for ordinary differential equations [6,27]. Cockburn and Shu [20,19,21] introduced the Runge–Kutta discontinuous Galerkin (RKDG) to solve first-order hyperbolic systems. The solution space of DG methods consists of piecewise continuous polynomial functions. As such, it can sharply capture discontinuities in the solution. They are also locally conservative, and can handle problems with complex geometries to high order. They have a simple communication pattern between elements with a common face, which is useful for parallel computation and adaptive methods, since it is easy to construct locally refined meshes with hanging nodes. Furthermore, they exhibit strong superconvergence that can be used to estimate the discretization error.

A posteriori error estimates are used to guide adaptive algorithms and stop the refinement process. An ideal estimate is (i) *asymptotically correct* in the sense that the error estimate in some norm approaches zero under mesh refinement at the same rate as the actual error and (ii) *computationally efficient* by requiring a small fraction of the solution cost. Several explicit *a posteriori* DG error estimates are known for hyperbolic problems [17,18] where upper bounds of the true error are derived in terms of local residuals and solution jumps. Goal oriented *a posteriori* error estimates have also been derived for hyperbolic systems [24,26]. Explicit error estimates are usually cheaper to use for steering adaptive refinement but can't be relied on to assess the solution quality since, in general, they fail to be asymptotically exact even for smooth solutions.

* Corresponding author.
E-mail address: adjerids@math.vt.edu (S. Adjerid).

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http://dx.doi.org/10.1016/j.apnum.2013.06.007

Adjerid et al. [6] developed the first asymptotically correct implicit *a posteriori* DG error estimates, based on superconvergence, for one-dimensional linear and nonlinear hyperbolic problems. Later, Adjerid and Massey [8,9] showed how to construct accurate error estimates for multi-dimensional scalar problems on rectangular meshes. They showed that the leading term of error is spanned by two $(p+1)$-degree Radau polynomials in the x and y directions, respectively. Krivodonova and Flaherty [25] showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$. They computed DG error estimates by solving local problems involving numerical fluxes, thus requiring information from neighboring *inflow* elements. Adjerid and Baccouch [2,3] investigated DG methods on structured and unstructured triangular meshes with several finite element spaces to compute accurate error estimates.

Several superconvergence results for DG methods are reported in the literature [2,6,9,7,14,16,15,13]. In [10], we proved that a projection of the DG solution is $O(h^{p+2})$ superconvergent at Radau points and constructed efficient *a posteriori* error estimates. However, our proofs were valid for special linear symmetric hyperbolic systems in two space dimensions satisfying the assumptions of Lemma 3.2 [10], for which either (i) at least one of the coefficient matrices $\mathbf{A}_1$ or $\mathbf{A}_2$, are invertible, or (ii) $\mathcal{N}(\mathbf{P}_{1,2}^t\mathbf{A}_2\mathbf{P}_{1,2}) = \{0\}$ or $\mathcal{N}(\mathbf{P}_{2,2}^t\mathbf{A}_1\mathbf{P}_{2,2}) = \{0\}$, where the $m\times(m-r)$ matrix $\mathbf{P}_{j,2}$, $j=1,2$, denotes the matrix of all $(m-r)$ orthogonal eigenvectors associated with the zero eigenvalue of $\mathbf{A}_j$. Unfortunately, many important hyperbolic systems such as the acoustic problem and Maxwell's equations do not satisfy these assumptions. In this manuscript, we show that the results in [10] hold for arbitrary linear symmetric hyperbolic systems with constant coefficient matrices $\mathbf{A}_1,\dots,\mathbf{A}_d$ in d space dimensions.

Thus, our *a posteriori* error estimates are implicit as they involve the solution of local problems for the error and are asymptotically exact provided the true solution is smooth enough. However solutions of hyperbolic systems may be discontinuous and thus our local theory does not hold on elements containing discontinuities or other singularities. Extensive computations suggest that the effectivity indices converge to unity under suitable adaptive mesh refinement even in the presence of discontinuities. Furthermore, our estimators have the following properties: (i) they yield accurate estimates of the true error in regions away from discontinuities, (ii) they underestimate the true error near singularities and in regions polluted by it, (iii) they may be used as error indicators for steering an adaptive mesh refinement algorithm. In some preliminary computations [5] we observed that an adaptive refinement strategy using an explicit error estimators for steering the adaptive refinement while using our error estimator for stopping the adaptive process can lead to much more efficient and robust algorithm than just using one estimator for both steering and stopping the adaptive process.

When used with a suitable adaptive algorithm our estimator tend to be asymptotically correct under adaptive mesh refinement in the presence of discontinuities. A possible explanation of this behavior is that the elements at the discontinuity, which are the source of high discretization and pollution errors, are refined such that their discretization and pollution errors are reduced to a harmless level. We also note that our error analysis does not include the effect of flux limiters and stabilization usually needed for high-order DG methods applied to hyperbolic systems with discontinuous solutions. This as well as the use of different numerical fluxes is currently under investigation.

This manuscript is organized as follows, in Section 2 we recall several results and preliminary results. In Section 3 we perform a local error analysis to investigate the asymptotic behavior of the local discretization error. In Section 4 we present our error estimation procedures and in Section 5 we present numerical results for several hyperbolic systems. We conclude and discuss our results in Section 6.

2. Problem formulation

Let d be the space dimension, $\mathbf{x}=(x_1,\dots,x_d)^t$ the space variable defined on a domain $\Omega=(0,1)^d\in\mathbb{R}^d$, and t the time variable defined on $[0,T]$.

Let $\mathbf{u}:[0,T]\times\Omega\to\mathbb{R}^m$ be the true solution of the linear symmetric hyperbolic system

$$\frac{\partial\mathbf{u}}{\partial t}+\sum_{i=1}^{d}\mathbf{A}_i\frac{\partial\mathbf{u}}{\partial x_i}=\mathbf{g}(t,\mathbf{x}),\quad \mathbf{x}\in\Omega,\ 0<t<T, \tag{2.1a}$$

with symmetric real constant coefficient matrices $\mathbf{A}_i\in\mathbb{R}^{m\times m}$, $1\leqslant i\leqslant d$, and subject to initial and boundary conditions

$$\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{2.1b}$$

$$\left(\sum_{i=1}^{d}(\nu_i\mathbf{A}_i)^-\right)\mathbf{u}(t,\mathbf{x})=\left(\sum_{i=1}^{d}(\nu_i\mathbf{A}_i)^-\right)\mathbf{u}_B(t,\mathbf{x}),\quad \mathbf{x}\in\partial\Omega,\ 0<t<T, \tag{2.1c}$$

where $\partial\Omega$ denotes the boundary of Ω and $\boldsymbol{\nu}$ denotes the unit outward normal on $\partial\Omega$. Since symmetric matrices are diagonalizable with real eigenvalues, we define $\mathbf{M}^{\pm}$ for a symmetric matrix $\mathbf{M}\in\mathbb{R}^{m\times m}$ by writing

$$\mathbf{M}=\mathbf{P}\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)\mathbf{P}^t,\quad \lambda_1,\dots,\lambda_m\in\mathbb{R}, \tag{2.2a}$$

$$\mathbf{M}^+=\mathbf{P}\,\mathrm{diag}\big(\max(\lambda_1,0),\dots,\max(\lambda_m,0)\big)\mathbf{P}^t, \tag{2.2b}$$

$$\mathbf{M}^- = \mathbf{P}\,\mathrm{diag}\big(\min(\lambda_1, 0), \ldots, \min(\lambda_m, 0)\big)\mathbf{P}^t, \tag{2.2c}$$

$$\mathrm{sgn}(\mathbf{M}) = \mathbf{P}\,\mathrm{diag}\big(\mathrm{sgn}(\lambda_1), \ldots, \mathrm{sgn}(\lambda_1)\big)\mathbf{P}^t, \tag{2.2d}$$

where $\mathrm{sgn}(x)$, $x \in \mathbb{R}$ denotes the standard *signum* function.

Using basic linear algebra, we prove that these matrices satisfy the following properties summarized in a lemma.

Lemma 2.1. *Let* $\mathbf{M}$ *be a real symmetric matrix and* $\mathbf{M}^+$ *and* $\mathbf{M}^-$ *as defined above. Then*

$$\mathcal{R}(\mathbf{M}) = \mathcal{N}(\mathbf{M})^\perp, \tag{2.3a}$$

$$\mathbf{M} = \mathbf{M}^+ + \mathbf{M}^-, \tag{2.3b}$$

$$\mathrm{sgn}(\mathbf{M}) \text{ is symmetric}, \tag{2.3c}$$

$$\mathbf{M}^+ \text{ and } -\mathbf{M}^- \text{ are symmetric positive semi-definite}, \tag{2.3d}$$

$$\mathcal{N}\big(\mathbf{M}^+ - \mathbf{M}^-\big) = \mathcal{N}\big(\mathrm{sgn}(\mathbf{M})\big) = \mathcal{N}(\mathbf{M}) \subseteq \mathcal{N}\big(\mathbf{M}^s\big), \quad s = +, -, \tag{2.3e}$$

$$\mathcal{R}\big(\mathbf{M}^s\big) \subseteq \mathcal{R}(\mathbf{M}) = \mathcal{R}\big(\mathrm{sgn}(\mathbf{M})\big), \quad s = +, -, \tag{2.3f}$$

$$\mathbf{M}^+\mathrm{sgn}(\mathbf{M}) = \mathbf{M}^+, \qquad \mathbf{M}^-\mathrm{sgn}(\mathbf{M}) = -\mathbf{M}^-. \tag{2.3g}$$

Now, let us partition the domain $\Omega = (0, 1)^d$ into a uniform mesh $\mathcal{T}_h$ consisting of N^d square elements of size $h = N^{-1}$ defined as

$$\mathcal{T}_h = \left\{ \prod_{i=1}^{d} (n_i h, n_i h + h)\colon\ 0 \leqslant n_i < N,\ 1 \leqslant i \leqslant d \right\}. \tag{2.4a}$$

Let $\mathcal{P}_p$, $p \geqslant 0$, denote the polynomial in $\mathbf{x}$ with coefficients in $\mathbb{R}^m$ of total degree at most $p+1$ and of degree at most p in each space variable $x_1, \ldots, x_d$, defined by

$$\mathcal{P}_p = \left\{ \sum_{\boldsymbol{\alpha}} \mathbf{c}_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}\colon\ \mathbf{c}_{\boldsymbol{\alpha}} \in \mathbb{R}^m,\ |\boldsymbol{\alpha}| \leqslant p+1,\ \alpha_i \leqslant p \right\}. \tag{2.4b}$$

Here, we use the finite element space

$$\mathcal{V}_p^h = \big\{ \mathbf{v}(t, \mathbf{x})\colon\ \mathbf{v}|_\omega \in \mathcal{P}_p,\ \omega \in \mathcal{T}_h,\ 0 \leqslant t \leqslant T \big\}. \tag{2.4c}$$

The weak formulation of (2.1a) is obtained by multiplying (2.1a) by a test function $\mathbf{v}$, integrating over an arbitrary element $\omega \in \mathcal{T}_h$, and applying Green's identity to write

$$\int_\omega \mathbf{v}^t \left(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{g} \right) d\mathbf{x} = \sum_{i=1}^{d} \int_\omega \frac{\partial \mathbf{v}^t}{\partial x_i} \mathbf{A}_i \mathbf{u}\, d\mathbf{x} - \int_{\partial\omega} \mathbf{v}^t \nu_i \mathbf{A}_i \mathbf{u}\, d\mathbf{s}, \quad \forall \omega \in \mathcal{T}_h,\ 0 < t < T, \tag{2.5}$$

where $\partial\omega$ denotes the boundary of ω and $\boldsymbol{\nu}$ its outward unit normal.

Since $\mathcal{V}_p^h$ allows discontinuities across element boundaries $\partial\omega$ of any element $\omega \in \mathcal{T}_h$, we define the traces of $\mathbf{u}_h \in \mathcal{V}_p^h$ on $\partial\omega$ as

$$\mathbf{u}_h^+(t, \mathbf{x}) = \lim_{\epsilon \to 0^+} \mathbf{u}_h(t, \mathbf{x} - \epsilon\boldsymbol{\nu}), \qquad \mathbf{u}_h^-(t, \mathbf{x}) = \lim_{\epsilon \to 0^+} \mathbf{u}_h(t, \mathbf{x} + \epsilon\boldsymbol{\nu}), \quad \mathbf{x} \in \partial\omega, \tag{2.6}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal on the boundary $\partial\omega$. We write $\mathbf{u}_h = \mathbf{u}_h^+$ whenever there is no confusion.

Applying the *Steger–Warming numerical flux* [29],

$$\mathbf{h}\big(\mathbf{u}_h^+, \mathbf{u}_h^-, \boldsymbol{\nu}\big) = \sum_{i=1}^{d} \big((\nu_i \mathbf{A}_i)^+ \mathbf{u}_h + (\nu_i \mathbf{A}_i)^- \mathbf{u}_h^- \big), \quad \mathbf{x} \in \partial\omega,\ \omega \in \mathcal{T}_h, \tag{2.7}$$

we complete the definition of the discontinuous Galerkin method which consists of finding $\mathbf{u}_h \in \mathcal{V}_p^h$ such that

$$\int_\omega \mathbf{v}^t \left(\frac{\partial \mathbf{u}_h}{\partial t} - \mathbf{g} \right) d\mathbf{x} = \sum_{i=1}^{d} \left(\int_\omega \frac{\partial \mathbf{v}^t}{\partial x_i} \mathbf{A}_i \mathbf{u}_h\, d\mathbf{x} - \int_{\partial\omega} \mathbf{v}^t \big((\nu_i \mathbf{A}_i)^+ \mathbf{u}_h + (\nu_i \mathbf{A}_i)^- \mathbf{u}_h^- \big)\, d\mathbf{s} \right), \quad \omega \in \mathcal{T}_h,\ \mathbf{v} \in \mathcal{P}_p,\ 0 < t < T. \tag{2.8}$$

This yields a system of ODEs which is integrated by the embedded Dormand–Prince method [22] with the temporal discretization error kept much smaller than the spatial error. However, for the purpose of analyzing the behavior of the spatial discretization error, we assume exact time integration.

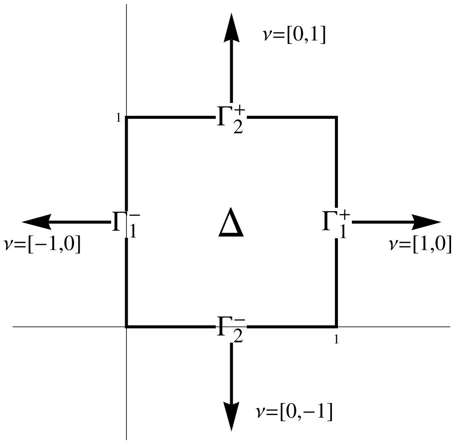

Fig. 1. The reference element $\Delta=(0,1)^d$ for $d=2$ with boundary $\varGamma$ and outward unit normal $\boldsymbol{\nu}$.

Initial and boundary conditions may be approximated using L^2-projections, however, solution superconvergence and *a posteriori* error estimation is polluted near $t=0$ but recovers rapidly with increasing t [10]. We expect a similar behavior of the error near the boundary if we do not use a consistent approximation of the boundary condition. In the remainder of this section we define consistent approximations of the initial and boundary conditions $\mathbf{u}_0$ and $\mathbf{u}_B$ by functions in $\mathcal{V}_p^h$ such that the approximation errors for the initial and boundary conditions have a similar behavior as the discontinuous Galerkin discretization error.

For simplicity, we only consider the approximation on the element $\omega=(0,h)^d$.

Let $\Delta=(0,1)^d$ denote the *reference element* and let $\varGamma$ denote its boundary.

We split $\varGamma=\bigcup_{i=1}^d \varGamma_i$, where

$$\varGamma_i=\varGamma_i^-\cup\varGamma_i^+,\qquad \varGamma_i^-=\{\boldsymbol{\xi}\in\Delta\colon \xi_i=0\},\qquad \varGamma_i^+=\{\boldsymbol{\xi}\in\Delta\colon \xi_i=1\},\quad 1\leqslant i\leqslant d. \tag{2.9a}$$

An illustration of Δ for $d=2$ is shown in Fig. 1.

For $\omega=(0,h)^d$ with boundary $\partial\omega$ there is an affine transformation $\mathbf{x}:\Delta\to\omega$, $\mathbf{x}(\boldsymbol{\xi})=h\boldsymbol{\xi}$. Then we split $\partial\omega=\bigcup_{i=1}^d\gamma_i$, where

$$\gamma_i=\gamma_i^-\cup\gamma_i^+,\qquad \gamma_i^\pm=\mathbf{x}\big(\varGamma_i^\pm\big),\quad 1\leqslant i\leqslant d. \tag{2.9b}$$

Let $L_p(\xi)$ denote the *Legendre polynomial of degree* p, as defined in [1], shifted to $[0,1]$. It is well known that $L_p(\xi)$ is orthogonal to all polynomials of degree not exceeding $p-1$ and satisfies

$$\int_0^1 L_p(\xi)L_q(\xi)\,d\xi=\frac{\delta_{pq}}{2p+1},\qquad \int_0^1 L_p(\xi)L'_{p+1}(\xi)\,d\xi=2, \tag{2.10}$$

where δ_{pq} is the Kronecker delta, which is equal to 1 if $p=q$ and 0 otherwise.

In order to obtain an error in the initial and boundary conditions consistent with the DG discretization error we define the operators π and π_i^s, $s=+,-$, $1\leqslant i\leqslant d$, respectively, to approximate the initial conditions on $\omega=(0,h)^d$ and the boundary conditions on $\gamma_i^s\cap\partial\Omega$.

For $\mathbf{v}\in[L^2(\omega)]^m$, if $\varPi\mathbf{v}$ denotes the L^2-projection of $\mathbf{v}$ onto $\mathcal{P}_p$, we define the approximation operator

$$\pi\mathbf{v}(\mathbf{x})=\varPi\mathbf{v}(\mathbf{x})+\sum_{i=1}^d L_p\left(\frac{x_i}{h}\right)\operatorname{sgn}(\mathbf{A}_i)\bar{\mathbf{c}}_i,\quad \bar{\mathbf{c}}_i=\frac{\int_\omega \mathbf{v}(\mathbf{x})L_{p+1}(\frac{x_i}{h})\,d\mathbf{x}}{\int_\omega L_{p+1}^2(\frac{x_i}{h})\,d\mathbf{x}}. \tag{2.11}$$

Similarly, for $\mathbf{v}\in[L^2(\gamma_i^s)]^m$, $s=+,-$, $1\leqslant i\leqslant d$, if $\varPi_i^s\mathbf{v}$ denotes the L^2-projection of $\mathbf{v}$ onto $\mathcal{P}_p|_{\gamma_i^s}$, we define

$$\pi_i^s\mathbf{v}(\mathbf{x})=\varPi_i^s\mathbf{v}+\sum_{j\in D(i)} L_p\left(\frac{x_i}{h}\right)\operatorname{sgn}(\mathbf{A}_i)\bar{\mathbf{c}}_{ij}^s, \tag{2.12a}$$

where $D(i)=\{1,2,\dots,d\}\setminus\{i\}$, and

$$\bar{\mathbf{c}}_{ij}^s=\frac{\int_{\gamma_i^s}\mathbf{v}(\mathbf{x})L_{p+1}(\frac{x_j}{h})\,d\mathbf{s}}{\int_{\gamma_i^s}L_{p+1}^2(\frac{x_j}{h})\,d\mathbf{s}},\qquad s=+,-,\ j\in D(i),\ 1\leqslant i\leqslant d. \tag{2.12b}$$

3. Local error analysis

In this section we perform a local error analysis on one element $\omega = (0,h)^d$ by writing the local error as a Maclaurin series and showing that its leading term can be expressed as a linear combination of Legendre polynomials of degree p and $p+1$. For special hyperbolic systems where the coefficient matrices are nonsingular we show that the leading term of the error is spanned by $(p+1)$-degree Radau polynomials.

Thus, let $\mathbf{u}_h \in \mathcal{P}_p$ satisfy the local DG formulation on $\omega = (0,h)^d$,

$$\int_\omega \mathbf{v}^t\left(\frac{\partial \mathbf{u}_h}{\partial t} - \mathbf{g}\right) d\mathbf{x} = \sum_{j=1}^d \left(\int_\omega \frac{\partial \mathbf{v}^t}{\partial x_j}\mathbf{A}_j\mathbf{u}_h\, d\mathbf{x} - \int_{\gamma_j} \mathbf{v}^t\left((\nu_j\mathbf{A}_j)^+\mathbf{u}_h + (\nu_j\mathbf{A}_j)^-\mathbf{u}_h^-\right) d\mathbf{s}\right), \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 0<t<T, \tag{3.1a}$$

subject to the initial and boundary conditions

$$\mathbf{u}_h(0,\mathbf{x}) = \pi\mathbf{u}_0(\mathbf{x}) \quad \text{or} \quad \mathbf{u}_h(0,\mathbf{x}) = \Pi\mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \omega, \tag{3.1b}$$

$$(\nu_i\mathbf{A}_i)^-\mathbf{u}_h^-(t,\mathbf{x}) = (\nu_i\mathbf{A}_i)^-\pi_i^s\mathbf{u}(t,\mathbf{x}), \quad \mathbf{x} \in \gamma_i^s,\ 0<t<T,\ s=+,-,\ 1 \leqslant i \leqslant d, \tag{3.1c}$$

where $\mathbf{u} = \mathbf{u}_B$ on the boundary of Ω.

3.1. Preliminary results

In this section we establish several bounds for the approximation errors of the initial and boundary conditions. We also state and prove several preliminary lemmas needed in our analysis.

For completeness we recall the following known Hölder's and inverse inequalities,

$$\|\mathbf{w}\|_{2,\omega} \leqslant |\omega|^{1/2}\|\mathbf{w}\|_{\infty,\omega}, \quad \forall \mathbf{w} \in L^\infty(\omega), \tag{3.2}$$

where $|\omega|$ denotes the area (volume) of ω, and the inverse inequality [12]

$$\|\mathbf{w}\|_{\infty,\omega} \leqslant C|\omega|^{-1/2}\|\mathbf{w}\|_{2,\omega}, \quad \forall \mathbf{w} \in \mathbb{P}_p, \tag{3.3}$$

where $C > 0$ and independent of $\mathbf{w}$.

Next, we state and establish bounds for the approximation errors from initial and boundary conditions.

Lemma 3.1. *Let $\omega = (0,h)^d$, $\mathbf{v}(\mathbf{x}) \in [C^{p+2}(\bar\omega)]^m$ and $\boldsymbol{\xi} = h^{-1}\mathbf{x}$. If $\Pi\mathbf{v}$ is the L^2-projection onto $\mathcal{P}_p$, $\pi\mathbf{v}$ is as defined in (2.11) on ω and $\pi_i^s\mathbf{v}$ as defined in (2.12), then there exists a positive constant C independent of h such that*

$$\left\|\mathbf{v}(\mathbf{x}) - \Pi\mathbf{v}(\mathbf{x}) - h^{p+1}\sum_{j=1}^d L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j\right\|_{\infty,\omega} \leqslant Ch^{p+2}, \tag{3.4a}$$

and on the boundary γ_i^s we have

$$\left\|\mathbf{v}(\mathbf{x}) - \pi_i^s\mathbf{v}(\mathbf{x}) - h^{p+1}\sum_{j\in D(i)}\left(L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j - L_p\left(\frac{x_j}{h}\right)\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_j\right)\right\|_{\infty,\gamma_i^s} \leqslant Ch^{p+2}, \quad s=+,-,\ 1\leqslant i\leqslant d, \tag{3.4b}$$

where

$$\mathbf{c}_j = \frac{1}{a_{p+1}}\frac{1}{(p+1)!}\frac{\partial^{p+1}\mathbf{v}(\mathbf{0})}{\partial x_j^{p+1}}, \quad 1 \leqslant j \leqslant d, \tag{3.4c}$$

and a_{p+1} denotes the coefficient of ξ^{p+1} in $L_{p+1}(\xi)$.

Proof. If $\mathcal{L}_{p+1}\mathbf{v}$ denotes the L^2-projection of $\mathbf{v}$ onto $[\mathbb{P}_{p+1}]^m$, one can write

$$\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x}) = \Pi\mathbf{v}(\mathbf{x}) + \sum_{i=1}^d L_{p+1}\left(\frac{x_i}{h}\right)\bar{\mathbf{c}}_i. \tag{3.5}$$

By the standard *a priori* error estimates we have

$$\left\|\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\right\|_{\infty,\omega} \leqslant Ch^{p+2}. \tag{3.6}$$

On the other hand, if $T_{p+1}\mathbf{v}$ denotes the $(p+1)$th-degree Taylor polynomial of $\mathbf{v}$ about $\mathbf{x}=\mathbf{0}$, the remainder can be bounded as

$$\left\|\mathbf{v}(\mathbf{x})-T_{p+1}\mathbf{v}(\mathbf{x})\right\|_{\infty,\omega}\leqslant Ch^{p+2}. \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$\left\|\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x})-T_{p+1}\mathbf{v}(\mathbf{x})\right\|_{\infty,\omega}\leqslant Ch^{p+2}. \tag{3.8}$$

Next, we show that

$$\left\|\bar{\mathbf{c}}_i-h^{p+1}\mathbf{c}_i\right\|\leqslant Ch^{p+2},\quad 1\leqslant i\leqslant d, \tag{3.9}$$

where $\|\mathbf{w}\|^2=\mathbf{w}^t\mathbf{w}$ denotes the l^2 vector norm.

By the definition of $\mathbf{c}_j$, $\bar{\mathbf{c}}_j$, $T_{p+1}\mathbf{v}$ and $\mathcal{L}_{p+1}\mathbf{v}$ we have that

$$\Pi\mathbf{v}(\mathbf{x})=\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x})-\sum_{j=1}^{d}L_{p+1}\left(\frac{x_j}{h}\right)\bar{\mathbf{c}}_j \tag{3.10}$$

and

$$S\mathbf{v}(\mathbf{x})=T_{p+1}\mathbf{v}(\mathbf{x})-h^{p+1}\sum_{j=1}^{d}L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j \tag{3.11}$$

belong to $\mathcal{P}_p$. Hence, they are both orthogonal to $(p+1)$-degree Legendre polynomials.

Multiplying $\Pi\mathbf{v}-S\mathbf{v}$ by $L_{p+1}(\frac{x_i}{h})$ and integrating over ω yields

$$\int_{\omega}\left(\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x})-T_{p+1}\mathbf{v}(\mathbf{x})-\sum_{j=1}^{d}L_{p+1}\left(\frac{x_j}{h}\right)\left(\bar{\mathbf{c}}_j-h^{p+1}\mathbf{c}_j\right)\right)L_{p+1}\left(\frac{x_i}{h}\right)d\mathbf{x}=\mathbf{0},\quad 1\leqslant i\leqslant d, \tag{3.12}$$

where we used the fact that $\int_{\omega}\mathbf{p}L_{p+1}(\frac{x_i}{h})\,d\mathbf{x}=\mathbf{0}$ for all $\mathbf{p}\in\mathcal{P}_p$.

Rearranging terms in (3.12) we write

$$\left(\bar{\mathbf{c}}_i-h^{p+1}\mathbf{c}_i\right)\left\|L_{p+1}\left(\frac{x_i}{h}\right)\right\|_{2,\omega}^2=\int_{\omega}\left(\mathcal{L}_{p+1}\mathbf{v}(\mathbf{x})-T_{p+1}\mathbf{v}(\mathbf{x})\right)L_{p+1}\left(\frac{x_i}{h}\right)d\mathbf{x}. \tag{3.13}$$

Taking the l^2 norm of (3.13), applying Schwarz's inequality, and dividing by $\|L_{p+1}(\frac{x_i}{h})\|_{2,\omega}^2$ yields

$$\left\|\bar{\mathbf{c}}_i-h^{p+1}\mathbf{c}_i\right\|\leqslant\frac{\|\mathcal{L}_{p+1}\mathbf{v}-T_{p+1}\mathbf{v}\|_{2,\omega}}{\|L_{p+1}(\frac{x_i}{h})\|_{2,\omega}}=|\omega|^{-\frac{1}{2}}(2p+3)\|\mathcal{L}_{p+1}\mathbf{v}-T_{p+1}\mathbf{v}\|_{2,\omega}, \tag{3.14}$$

where $|\omega|$ denotes the area (volume) of ω.

Applying Hölder's inequality (3.2) to (3.14) and using (3.8), we obtain

$$\left\|\bar{\mathbf{c}}_i-h^{p+1}\mathbf{c}_i\right\|\leqslant(2p+3)\|\mathcal{L}_{p+1}\mathbf{v}-T_{p+1}\mathbf{v}\|_{\infty,\omega}\leqslant Ch^{p+2},\quad 1\leqslant i\leqslant d, \tag{3.15}$$

which establishes (3.9).

Combining (3.6) and (3.9) yields

$$\begin{aligned}&\left\|\mathbf{v}(\mathbf{x})-\Pi\mathbf{v}(\mathbf{x})-h^{p+1}\sum_{i=1}^{d}L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{c}_i\right\|_{\infty,\omega}\\&\quad\leqslant\left\|\mathbf{v}(\mathbf{x})-\Pi\mathbf{v}(\mathbf{x})-\sum_{i=1}^{d}L_{p+1}\left(\frac{x_i}{h}\right)\bar{\mathbf{c}}_i\right\|_{\infty,\omega}+\sum_{i=1}^{d}\left\|\bar{\mathbf{c}}_i-h^{p+1}\mathbf{c}_i\right\|\leqslant Ch^{p+2},\end{aligned}$$

which proves (3.4a).

Following the same line of reasoning we establish (3.4b). □

In the next lemmas we establish useful properties of the subspace $\bar{\mathcal{P}}_p\subseteq\mathcal{P}_p$, defined by

$$\bar{\mathcal{P}}_p=\left\{\mathbf{v}(\boldsymbol{\xi})\in\mathcal{P}_p\colon\ \sum_{i=1}^{d}\mathbf{A}_i\frac{\partial\mathbf{v}}{\partial\xi_i}=\mathbf{0}\text{ on }\Delta,\ \mathbf{A}_i\mathbf{v}=\mathbf{0}\text{ on }\Gamma_i,\ 1\leqslant i\leqslant d\right\}, \tag{3.16a}$$

and the orthogonal complement of $\bar{\mathcal{P}}_p$ in $[L^2(\Delta)]^m$, defined by

$$\bar{\mathcal{P}}_p^{\perp} = \left\{ \mathbf{w}(\boldsymbol{\xi}) \in \left[L^2(\Delta)\right]^m \colon \int_{\Delta} \mathbf{v}^t \mathbf{w}\, d\boldsymbol{\xi} = 0,\ \forall \mathbf{v} \in \bar{\mathcal{P}}_p \right\}. \tag{3.16b}$$

Let $\Gamma_i(a)$ be a hyperplane through $\bar{\Delta}$, defined by

$$\Gamma_i(a) = \{\boldsymbol{\xi} \in \bar{\Delta} \colon \xi_i = a\}, \quad a \in [0,1]. \tag{3.17}$$

Note that $\Gamma_i(0) = \Gamma_i^-$ and $\Gamma_i(1) = \Gamma_i^+$, as defined in (2.9a).

We can show the following properties of $\bar{\mathcal{P}}_p$:

Lemma 3.2. *For all integrable functions* $\mathbf{f} : (0,1) \to \mathbb{R}^m$ *we have*

$$\int_{\Delta} \mathbf{v}^t \mathbf{A}_i \mathbf{f}(\xi_i)\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 1 \leqslant i \leqslant d. \tag{3.18}$$

Proof. Let $1 \leqslant i \leqslant d$ and $\mathbf{v} \in \bar{\mathcal{P}}_p$ and define the function

$$\mathbf{h}(\xi_i) = \int_{\Gamma_i(\xi_i)} \mathbf{A}_i \mathbf{v}\, d\boldsymbol{\sigma}, \quad \xi_i \in [0,1]. \tag{3.19}$$

By the definition of $\bar{\mathcal{P}}_p$, $\mathbf{A}_j \mathbf{v} = \mathbf{0}$ on Γ_j for all $1 \leqslant j \leqslant d$. The divergence theorem yields

$$\int_{\Gamma_i(\xi_i)} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j}\, d\boldsymbol{\sigma} = \int_{\Gamma_j^+ \cap \Gamma_i(\xi_i)} \mathbf{A}_j \mathbf{v}\, d\boldsymbol{\sigma} - \int_{\Gamma_j^- \cap \Gamma_i(\xi_i)} \mathbf{A}_j \mathbf{v}\, d\boldsymbol{\sigma} = \mathbf{0}, \quad j \in D(i),\ \xi_i \in [0,1]. \tag{3.20}$$

We note that by the definition of $\bar{\mathcal{P}}_p$, $\sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j} = \mathbf{0}$ on ω, which combined with (3.20) yields

$$\frac{d}{d\xi_i} \mathbf{h}(\xi_i) = \int_{\Gamma_i(\xi_i)} \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \xi_i}\, d\boldsymbol{\sigma} = -\sum_{j \in D(i)} \int_{\Gamma_i(\xi_i)} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j}\, d\boldsymbol{\sigma} = \mathbf{0}, \quad \xi_i \in (0,1). \tag{3.21}$$

Since $\mathbf{A}_i \mathbf{v} = \mathbf{0}$ on Γ_i, we obtain

$$\mathbf{h}(0) = \int_{\Gamma_i^-} \mathbf{A}_i \mathbf{v}\, d\boldsymbol{\sigma} = \mathbf{0}, \tag{3.22}$$

which together with the Fundamental Theorem of Calculus and (3.21) yields

$$\mathbf{h}(\xi_i) = \mathbf{h}(0) + \int_0^{\xi_i} \frac{d}{d\hat{\xi}_i} \mathbf{h}(\hat{\xi}_i)\, d\hat{\xi}_i = \mathbf{0}, \quad \xi_i \in [0,1]. \tag{3.23}$$

Since $\mathbf{f}(\xi_i)$ is constant on $\Gamma_i(\xi_i)$, and $\mathbf{A}_i$ is symmetric, we obtain

$$\int_{\Gamma_i(\xi_i)} \mathbf{v}^t \mathbf{A}_i \mathbf{f}(\xi_i)\, d\boldsymbol{\sigma} = \int_{\Gamma_i(\xi_i)} \mathbf{f}^t(\xi_i) \mathbf{A}_i \mathbf{v}\, d\boldsymbol{\sigma} = \mathbf{f}^t(\xi_i) \mathbf{h}(\xi_i) = 0, \quad \xi_i \in [0,1]. \tag{3.24}$$

Integrating (3.24) over ξ_i from 0 to 1 yields (3.18). □

Lemma 3.3. *Let* $\mathbf{d} \in \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k)$. *Then*

$$\int_{\Delta} \mathbf{v}^t \mathbf{d} L_p(\xi_i)\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 1 \leqslant i \leqslant d. \tag{3.25}$$

Proof. Since $\mathbf{d} \in \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k)$, it can be written as

$$\mathbf{d} = \sum_{k=1}^{d} \mathbf{A}_k \mathbf{d}_k. \tag{3.26}$$

By Lemma 3.2 we know that

$$\int_{\Delta} \mathbf{v}^t \mathbf{A}_i \mathbf{d}_i L_p(\xi_i)\, d\boldsymbol{\xi} = 0, \quad 1 \leqslant i \leqslant d. \tag{3.27}$$

Now, we show

$$\int_{\Delta} \mathbf{v}^t \mathbf{A}_k \mathbf{d}_k L_p(\xi_i)\, d\boldsymbol{\xi} = 0, \quad k \in D(i),\ 1 \leqslant i \leqslant d. \tag{3.28}$$

First, we define the auxiliary function

$$\tilde{\mathbf{h}}_k(\xi_k) = \int_{\Gamma_k(\xi_k)} \mathbf{A}_k \mathbf{v} L_p(\xi_i)\, d\boldsymbol{\sigma}, \quad \xi_k \in [0,1]. \tag{3.29}$$

By the definition of $\bar{\mathcal{P}}_p$, $\mathbf{A}_k \frac{\partial \mathbf{v}}{\partial \xi_k} = -\sum_{j \in D(k)} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j}$, which yields

$$\frac{d}{d\xi_k} \tilde{\mathbf{h}}_k(\xi_k) = \int_{\Gamma_k(\xi_k)} \mathbf{A}_k \frac{\partial \mathbf{v}}{\partial \xi_k} L_p(\xi_i)\, d\boldsymbol{\sigma} = -\sum_{j \in D(k)} \int_{\Gamma_k(\xi_k)} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j} L_p(\xi_i)\, d\boldsymbol{\sigma}, \quad \xi_k \in (0,1). \tag{3.30}$$

Next, we show $\frac{d}{d\xi_k} \tilde{\mathbf{h}}_k(\xi_k) = \mathbf{0}$, $\xi_k \in [0,1]$.

Since $\mathbf{v} \in \bar{\mathcal{P}}_p \subset \mathcal{P}_p$, we obtain by the orthogonality properties of Legendre polynomials that

$$\int_{\Gamma_k(\xi_k)} \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \xi_i} L_p(\xi_i)\, d\boldsymbol{\sigma} = \mathbf{0}, \quad \xi_k \in [0,1]. \tag{3.31}$$

By the definition of $\bar{\mathcal{P}}_p$, $\mathbf{A}_j \mathbf{v} = \mathbf{0}$ on Γ_j for all $1 \leqslant j \leqslant d$, which together with the divergence theorem yields

$$\int_{\Gamma_k(\xi_k)} \mathbf{A}_j \frac{\partial \mathbf{v}}{\partial \xi_j} L_p(\xi_i)\, d\boldsymbol{\sigma} = \int_{\Gamma_k(\xi_k) \cap \Gamma_j} \nu_j \mathbf{A}_j \mathbf{v} L_p(\xi_i)\, d\boldsymbol{\sigma} = \mathbf{0}, \quad j \in D(k) \cap D(i),\ \xi_k \in [0,1], \tag{3.32}$$

where Γ_j is defined in (2.9a).

Substituting (3.31) and (3.32) into (3.30) yields

$$\frac{d}{d\xi_k} \tilde{\mathbf{h}}_k(\xi_k) = \mathbf{0}, \quad \xi_k \in (0,1). \tag{3.33}$$

Since $\mathbf{A}_k \mathbf{v} = \mathbf{0}$ on Γ_k and Γ_k^-, we obtain

$$\tilde{\mathbf{h}}_k(0) = \int_{\Gamma_k^-} \mathbf{A}_k \mathbf{v} L_p(\xi_i)\, d\boldsymbol{\sigma} = 0, \tag{3.34}$$

which together with the Fundamental Theorem of Calculus and (3.33) yields

$$\tilde{\mathbf{h}}_k(\xi_k) = \mathbf{0}, \quad \xi_k \in [0,1]. \tag{3.35}$$

Since $\mathbf{d}_k$ is constant and $\mathbf{A}_k$ is symmetric, we obtain

$$\int_{\Gamma_k(\xi_k)} \mathbf{v}^t \mathbf{A}_k \mathbf{d}_k L_p(\xi_i)\, d\boldsymbol{\sigma} = \int_{\Gamma_k(\xi_k)} \mathbf{d}_k^t \mathbf{A}_k \mathbf{v} L_p(\xi_i)\, d\boldsymbol{\sigma} = \mathbf{d}_k^t \tilde{\mathbf{h}}_k(\xi_k) = 0, \quad \xi_k \in [0,1]. \tag{3.36}$$

Integrating (3.36) from $\xi_k = 0$ to 1 yields (3.28).

Combining (3.27) and (3.28) yields

$$\int_\Delta \mathbf{v}^t \mathbf{d} L_p(\xi_i)\, d\boldsymbol{\xi} = \sum_{k=1}^{d} \int_\Delta \mathbf{v}^t \mathbf{A}_k \mathbf{d}_k L_p(\xi_i)\, d\boldsymbol{\xi} = 0. \tag{3.37}$$

This concludes the proof. □

Lemmas 3.2 and 3.3 can be used to show the following orthogonality condition for $\mathbf{v} \in \bar{\mathcal{P}}_p$.

Lemma 3.4. *Let* $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{d} \in \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k)$. *Then*

$$\int_\Delta \mathbf{v}^t \big(L_{p+1}(\xi_i)\mathbf{c} - L_p(\xi_i)\big(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c} + \mathbf{d}\big)\big)\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 1 \leqslant i \leqslant d. \tag{3.38}$$

Proof. By [23], $\mathbf{A}_i\mathbf{A}_i^\dagger$ is the orthogonal projection onto $\mathcal{R}(\mathbf{A}_i)$, where $\mathbf{A}_i^\dagger$ denotes the pseudoinverse of $\mathbf{A}_i$. Since $\mathcal{R}(\mathbf{A}_i) = \mathcal{R}(\operatorname{sgn}(\mathbf{A}_i))$ by property (2.3f), we obtain

$$\operatorname{sgn}(\mathbf{A}_i) = \mathbf{A}_i\mathbf{A}_i^\dagger \operatorname{sgn}(\mathbf{A}_i), \quad 1 \leqslant i \leqslant d, \tag{3.39}$$

which, when combined with Lemma 3.2, yields

$$\int_\Delta \mathbf{v}^t \operatorname{sgn}(\mathbf{A}_i)\mathbf{c} L_p(\xi_i)\, d\boldsymbol{\xi} = \int_\Delta \mathbf{v}^t \mathbf{A}_i \big(\mathbf{A}_i^\dagger \operatorname{sgn}(\mathbf{A}_i)\mathbf{c} L_p(\xi_i)\big)\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 1 \leqslant i \leqslant d. \tag{3.40}$$

Lemma 3.3 and the orthogonality properties (2.10) combined yield

$$\int_\Delta \mathbf{v}^t \big(L_{p+1}(\xi_i)\mathbf{c} - L_p(\xi_i)\mathbf{d}\big)\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 1 \leqslant i \leqslant d. \tag{3.41}$$

Adding (3.40) and (3.41) yields (3.38). □

We also need the following lemma:

Lemma 3.5. *If* $\mathbf{q} \in \mathcal{P}_p$ *satisfies*

$$\sum_{i=1}^{d} \left(\int_\Delta \frac{\partial \mathbf{v}^t}{\partial \xi_i} \mathbf{A}_i \mathbf{q}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \mathbf{q}\, d\boldsymbol{\sigma} \right) = 0, \quad \mathbf{v} \in \mathcal{P}_p, \tag{3.42}$$

then $\mathbf{q} \in \bar{\mathcal{P}}_p$.

Proof. First, we integrate (3.42) by parts to obtain

$$\sum_{i=1}^{d} \left(-\int_\Delta \mathbf{v}^t \mathbf{A}_i \frac{\partial \mathbf{q}}{\partial \xi_i}\, d\xi + \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^- \mathbf{q}\, d\boldsymbol{\sigma} \right) = 0, \quad \mathbf{v} \in \mathcal{P}_p. \tag{3.43}$$

Adding (3.42) to (3.43), testing against $\mathbf{v} = -\mathbf{q}$, and using the symmetry of $\mathbf{A}_i$, $1 \leqslant i \leqslant d$, we obtain

$$\sum_{i=1}^{d} \int_{\Gamma_i} \mathbf{q}^t \big((\nu_i \mathbf{A}_i)^+ - (\nu_i \mathbf{A}_i)^-\big)\mathbf{q}\, d\boldsymbol{\sigma} = \sum_{i=1}^{d} \int_{\Gamma_i} \mathbf{q}^t \big(\mathbf{A}_i^+ - \mathbf{A}_i^-\big)\mathbf{q}\, d\boldsymbol{\sigma} = 0. \tag{3.44}$$

$(\mathbf{A}_i^+ - \mathbf{A}_i^-)$ is symmetric positive semi-definite by Lemma 2.1, and therefore admits a Cholesky factorization $(\mathbf{A}_i^+ - \mathbf{A}_i^-) = \mathbf{L}_i^t \mathbf{L}_i$. Hence (3.44) can be written as

$$\sum_{i=1}^{d} \int_{\Gamma_i} \|\mathbf{L}_i \mathbf{q}\|^2\, d\boldsymbol{\sigma} = 0. \tag{3.45}$$

Therefore, $\mathbf{L}_i\mathbf{q} = \mathbf{0}$ on Γ_i, which yields

$$\mathbf{L}_i^t(\mathbf{L}_i\mathbf{q}) = \left(\mathbf{A}_i^+ - \mathbf{A}_i^-\right)\mathbf{q} = \mathbf{0}, \quad \boldsymbol{\xi} \in \Gamma_i,\ 1 \leqslant i \leqslant d, \tag{3.46}$$

which combined with property (2.3e) leads to

$$\mathbf{A}_i^s\mathbf{v} = \mathbf{0}, \quad \boldsymbol{\xi} \in \Gamma_i,\ s = +, -,\ 1 \leqslant i \leqslant d. \tag{3.47}$$

By (3.47), the boundary integral in (3.43) vanishes. Thus, (3.43) yields for $\mathbf{v} = \sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{q}}{\partial \xi_i}$

$$-\int_\Delta \left\| \sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{q}}{\partial \xi_i} \right\|^2 d\boldsymbol{\xi} = 0, \tag{3.48}$$

which in turn yields

$$\sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{q}}{\partial \xi_i} = \mathbf{0}, \quad \boldsymbol{\xi} \in \Delta. \tag{3.49}$$

Combining (3.47) and (3.49) proves the lemma. □

3.2. Asymptotic behavior of the local discretization error

Now we are ready to state a theorem for the local discretization error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$.

Theorem 3.6. *Let $\mathbf{u} \in [C^2([0,T], C^{p+2}(\bar{\omega}))]^m$ be the solution of (2.1) and let $\mathbf{u}_h \in \mathcal{P}_p$ satisfy (3.1). Then the local finite element error on ω, at $t = \mathcal{O}(1)$ and for $p \geqslant 1$, can be written as*

$$\mathbf{e}(t, h\boldsymbol{\xi}) = h^{p+1} \sum_{i=1}^d \mathbf{r}_i(t, h\xi_i) + \mathcal{O}\left(h^{p+2}\right), \quad \boldsymbol{\xi} \in \Delta, \tag{3.50a}$$

where

$$\mathbf{r}_i(t, h\xi_i) = L_{p+1}(\xi_i)\mathbf{c}_i(t) - L_p(\xi_i)\left(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i(t) + \mathbf{d}_i(t)\right), \quad 1 \leqslant i \leqslant d, \tag{3.50b}$$

with

$$\mathbf{c}_i(t) = \frac{1}{a_{p+1}} \frac{1}{(p+1)!} \frac{\partial^{p+1}\mathbf{u}(t, \mathbf{0})}{\partial x_i^{p+1}}, \qquad \mathbf{d}_i(t) \in \mathcal{N}(\mathbf{A}_i) \cap \bigoplus_{k=1}^d \mathcal{R}(\mathbf{A}_k). \tag{3.50c}$$

Proof. First, we derive the *orthogonality condition* for the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. By (2.1a), $\mathbf{u}$ satisfies

$$\int_\omega \mathbf{v}^t\left(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{g}\right) d\mathbf{x} = \sum_{i=1}^d \left(\int_\omega \frac{\partial \mathbf{v}^t}{\partial x_i} \mathbf{A}_i \mathbf{u}\, d\mathbf{x} - \int_{\gamma_i} \mathbf{v}^t \nu_i \mathbf{A}_i \mathbf{u}\, d\mathbf{s} \right), \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 0 < t < T. \tag{3.51}$$

Subtracting (3.1) from (3.51) we obtain

$$\int_\omega \mathbf{v}^t \frac{\partial \mathbf{e}}{\partial t} d\mathbf{x} = \sum_{i=1}^d \left(\int_\omega \frac{\partial \mathbf{v}^t}{\partial x_i} \mathbf{A}_i \mathbf{e}\, d\mathbf{x} - \int_{\gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \mathbf{e} + \mathbf{v}^t (\nu_i \mathbf{A}_i)^- \mathbf{e}^-\, d\mathbf{s} \right), \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 0 < t < T. \tag{3.52}$$

Apply the scalings $\tau = T^{-1}t$ and $\boldsymbol{\xi} = h^{-1}\mathbf{x}$ and write $\hat{\mathbf{e}}(\tau, \boldsymbol{\xi}) = \mathbf{e}(T\tau, h\boldsymbol{\xi})$ to obtain the orthogonality condition

$$\frac{h}{T} \int_\Delta \mathbf{v}^t \frac{\partial \hat{\mathbf{e}}}{\partial \tau} d\boldsymbol{\xi} = \sum_{i=1}^d \left(\int_\Delta \frac{\partial \mathbf{v}^t}{\partial \xi_i} \mathbf{A}_i \hat{\mathbf{e}}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \hat{\mathbf{e}} + \mathbf{v}^t (\nu_i \mathbf{A}_i)^- \hat{\mathbf{e}}^-\, d\boldsymbol{\sigma} \right), \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 0 < \tau < 1. \tag{3.53}$$

Now note that, since $\mathcal{P}_p$ is a subspace of $[L^2(\Delta)]^m$, we can split $\hat{\mathbf{e}}$ by

$$\hat{\mathbf{e}} = \bar{\mathbf{e}} + \bar{\mathbf{e}}^\perp, \quad \bar{\mathbf{e}} \in \bar{\mathcal{P}}_p,\ \bar{\mathbf{e}}^\perp \in \bar{\mathcal{P}}_p^\perp. \tag{3.54}$$

First, we will show that

$$\bar{\mathbf{e}}(\tau, \boldsymbol{\xi}) = \mathcal{O}\left(h^{p+2}\right), \quad \boldsymbol{\xi} \in \Delta,\ 0 \leqslant \tau \leqslant 1. \tag{3.55}$$

Since $\bar{\mathcal{P}}_p$ is a finite-dimensional vector space and $\bar{\mathbf{e}} \in \bar{\mathcal{P}}_p$, we have

$$\frac{\partial \bar{\mathbf{e}}}{\partial \tau}(\tau, \boldsymbol{\xi}) = \lim_{h \to 0} \frac{\bar{\mathbf{e}}(\tau + h, \boldsymbol{\xi}) - \bar{\mathbf{e}}(\tau, \boldsymbol{\xi})}{h} \in \bar{\mathcal{P}}_p, \tag{3.56a}$$

$$\frac{\partial \bar{\mathbf{e}}^{\perp}}{\partial \tau}(\tau, \xi) = \lim_{h \to 0} \frac{\bar{\mathbf{e}}^{\perp}(\tau + h, \boldsymbol{\xi}) - \bar{\mathbf{e}}^{\perp}(\tau, \boldsymbol{\xi})}{h} \in \bar{\mathcal{P}}_p^{\perp}, \quad \boldsymbol{\xi} \in \Delta,\ 0 < \tau < 1. \tag{3.56b}$$

By the definition of $\bar{\mathcal{P}}_p$ in (3.16) and the symmetry of $\mathbf{A}_i$, $\mathbf{A}_i^+$, and $\mathbf{A}_i^-$, $1 \leqslant i \leqslant d$, (3.53) yields for $\mathbf{v} \in \bar{\mathcal{P}}_p$

$$\begin{aligned} \frac{h}{T} \int_{\Delta} \mathbf{v}^t \frac{\partial \hat{\mathbf{e}}}{\partial \tau}\, d\boldsymbol{\xi} &= \sum_{i=1}^{d} \left(\int_{\Delta} \Big(\mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \xi_i}\Big)^t \hat{\mathbf{e}}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \big((\nu_i \mathbf{A}_i)^+ \mathbf{v}\big)^t \hat{\mathbf{e}} + \big((\nu_i \mathbf{A}_i)^- \mathbf{v}\big)^t \hat{\mathbf{e}}^-\, d\boldsymbol{\sigma} \right) \\ &= 0, \quad \forall \mathbf{v} \in \bar{\mathcal{P}}_p,\ 0 < \tau < 1. \end{aligned} \tag{3.57}$$

Thus, $\frac{\partial \hat{\mathbf{e}}}{\partial \tau} \in \bar{\mathcal{P}}_p^{\perp}$, which combined with (3.54) and (3.56b) yields

$$\frac{\partial \bar{\mathbf{e}}}{\partial \tau} = \frac{\partial \hat{\mathbf{e}}}{\partial \tau} - \frac{\partial \bar{\mathbf{e}}^{\perp}}{\partial \tau} \in \bar{\mathcal{P}}_p^{\perp}, \quad 0 < \tau < 1. \tag{3.58}$$

Then (3.58) and (3.56a) together yield

$$\frac{\partial \bar{\mathbf{e}}}{\partial \tau}(\tau, \boldsymbol{\xi}) = \mathbf{0}, \quad \boldsymbol{\xi} \in \Delta,\ 0 < \tau < 1. \tag{3.59}$$

By Lemma 3.1, the initial conditions satisfy either

$$\hat{\mathbf{e}}(0, \boldsymbol{\xi}) = \mathbf{u}_0(h\xi) - \Pi \mathbf{u}_0(h\boldsymbol{\xi}) = h^{p+1} \sum_{i=1}^{d} L_{p+1}(\xi_i)\mathbf{c}_i + \mathcal{O}\big(h^{p+2}\big), \quad \boldsymbol{\xi} \in \Delta, \tag{3.60a}$$

or

$$\hat{\mathbf{e}}(0, \boldsymbol{\xi}) = \mathbf{u}_0(h\xi) - \pi \mathbf{u}_0(h\boldsymbol{\xi}) = h^{p+1} \sum_{i=1}^{d} \mathbf{r}_i(0, \xi_i) + \mathcal{O}\big(h^{p+2}\big), \quad \boldsymbol{\xi} \in \Delta. \tag{3.60b}$$

By the orthogonality properties (2.10) and Lemma 3.4, we have

$$L_{p+1}(\xi_i)\mathbf{c}_i \in \bar{\mathcal{P}}_p^{\perp}, \qquad \mathbf{r}_i(0, \xi_i) \in \bar{\mathcal{P}}_p^{\perp}, \quad 1 \leqslant i \leqslant d, \tag{3.61}$$

which, when combined with (3.60), yields

$$\bar{\mathbf{e}}(0, \boldsymbol{\xi}) = \mathcal{O}\big(h^{p+2}\big), \quad \boldsymbol{\xi} \in \Delta. \tag{3.62}$$

By the Fundamental Theorem of Calculus, (3.62) and (3.59) yield (3.55).

In the remainder of the proof, we investigate the asymptotic behavior of $\bar{\mathbf{e}}^{\perp}$. We write the Maclaurin series of $\hat{\mathbf{e}}$ with respect to the mesh parameter h as

$$\hat{\mathbf{e}}(\tau, \boldsymbol{\xi}) = \sum_{k=0}^{p+1} h^k \mathbf{q}_k(\tau, \boldsymbol{\xi}) + \mathcal{O}\big(h^{p+2}\big), \quad \boldsymbol{\xi} \in \Delta,\ 0 < \tau < 1, \tag{3.63}$$

where, since $\mathbf{u}_h$ is a function of $T\tau$, $h\boldsymbol{\xi}$, and h,

$$\mathbf{q}_k(\tau, \boldsymbol{\xi}) = \frac{1}{k!} \frac{d^k(\mathbf{u}(T\tau, h\xi) - \mathbf{u}_h(T\tau, h\xi, h))}{dh^k}\bigg|_{h=0}. \tag{3.64}$$

We write the Maclaurin series of $\bar{\mathbf{e}}^{\perp} \in \bar{\mathcal{P}}_p^{\perp}$ with respect to the mesh parameter h as

$$\bar{\mathbf{e}}^{\perp}(\tau, \boldsymbol{\xi}) = \sum_{k=0}^{\infty} h^k \tilde{\mathbf{q}}_k(\tau, \boldsymbol{\xi}), \quad \tilde{\mathbf{q}}_k \in \bar{\mathcal{P}}_p^{\perp},\ \boldsymbol{\xi} \in \Delta,\ 0 < \tau < 1. \tag{3.65}$$

By (3.54) and (3.62), $\hat{\mathbf{e}} = \bar{\mathbf{e}}^{\perp} + \mathcal{O}(h^{p+2})$, thus subtracting (3.63) from (3.65) and setting all terms having the same power of h equal yields

$$\mathbf{q}_k = \tilde{\mathbf{q}}_k \in \bar{\mathcal{P}}_p^{\perp}, \quad 0 \leqslant k \leqslant p + 1. \tag{3.66}$$

Let $\hat{\mathbf{r}}_i(\tau,\xi_i)=\mathbf{r}_i(T\tau,h\xi_i)$, $1\leqslant i\leqslant d$. By Lemma 3.1, the boundary conditions satisfy

$$\hat{\mathbf{e}}^-(\tau,\boldsymbol{\xi})=\mathbf{u}(t,h\boldsymbol{\xi})-\mathbf{u}_h^-(t,h\boldsymbol{\xi})=h^{p+1}\sum_{j\in D(i)}\hat{\mathbf{r}}_j(\tau,\xi_j)+\mathcal{O}\big(h^{p+2}\big),\quad \boldsymbol{\xi}\in\Gamma_i,\ 1\leqslant i\leqslant d. \tag{3.67}$$

Substituting (3.63) and (3.67) in (3.53) yields

$$\sum_{k=0}^{p+1}h^k\left(\frac{h}{T}\int_\Delta \mathbf{v}^t\frac{\partial\mathbf{q_k}}{\partial\tau}\,d\boldsymbol{\xi}-\sum_{i=1}^d\left(\int_\Delta\frac{\partial\mathbf{v}^t}{\partial\xi_i}\mathbf{A}_i\mathbf{q}_k\,d\boldsymbol{\xi}+\int_{\Gamma_i}\mathbf{v}^t(\nu_i\mathbf{A}_i)^+\mathbf{q}_k\,d\boldsymbol{\sigma}\right)\right)$$
$$=-h^{p+1}\sum_{i=1}^d\left(\int_{\Gamma_i}\mathbf{v}^t(\nu_i\mathbf{A}_i)^-\sum_{j\in D(i)}\hat{\mathbf{r}}_j\,d\sigma\right)+\mathcal{O}\big(h^{p+2}\big),\quad \mathbf{v}\in\mathcal{P}_p. \tag{3.68}$$

Now assume that $T=\mathcal{O}(1)$ and set to zero all terms in (3.68) having the same power of h.

The $\mathcal{O}(1)$ term $\mathbf{q}_0$ satisfies the orthogonality condition

$$\sum_{i=1}^d\left(\int_\Delta\frac{\partial\mathbf{v}^t}{\partial\xi_i}\mathbf{A}_i\mathbf{q}_0\,d\boldsymbol{\xi}-\int_{\Gamma_i}\mathbf{v}^t(\nu_i\mathbf{A}_i)^+\mathbf{q}_0\,d\boldsymbol{\sigma}\right)=0,\quad \mathbf{v}\in\mathcal{P}_p. \tag{3.69}$$

Lemma 3.5 yields $\mathbf{q}_0\in\bar{\mathcal{P}}_p$, which combined with (3.66) shows that $\mathbf{q}_0=\mathbf{0}$ on Δ.

Assume that $\mathbf{q}_j=\mathbf{0}$ for all $0\leqslant j\leqslant k-1$, where $k\leqslant p$. Thus, the $\mathcal{O}(h^k)$ term is written as

$$\sum_{i=1}^d\left(\int_\Delta\frac{\partial\mathbf{v}^t}{\partial\xi_i}\mathbf{A}_i\mathbf{q}_k\,d\boldsymbol{\xi}-\int_{\Gamma_i}\mathbf{v}^t(\nu_i\mathbf{A}_i)^+\mathbf{q}_k\,d\boldsymbol{\sigma}\right)=0,\quad \mathbf{v}\in\mathcal{P}_p. \tag{3.70}$$

Lemma 3.5 yields $\mathbf{q}_k\in\bar{\mathcal{P}}_p$, which combined with (3.66) shows that $\mathbf{q}_k=\mathbf{0}$ on Δ for $0\leqslant k\leqslant p$.

The $\mathcal{O}(h^{p+1})$ term satisfies the orthogonality condition

$$\sum_{i=1}^d\left(\int_\Delta\frac{\partial\mathbf{v}^t}{\partial\xi_i}\mathbf{A}_i\mathbf{q}_{p+1}\,d\boldsymbol{\xi}-\int_{\Gamma_i}\mathbf{v}^t(\nu_i\mathbf{A}_i)^+\mathbf{q}_{p+1}-\mathbf{v}^t(\nu_i\mathbf{A}_i)^-\left(\sum_{j\in D(i)}\hat{\mathbf{r}}_j\right)d\boldsymbol{\sigma}\right)=0,\quad \forall\mathbf{v}\in\mathcal{P}_p. \tag{3.71}$$

First, we show $\mathbf{q}_{p+1}=\sum_{i=1}^d\hat{\mathbf{r}}_i+\mathbf{p}$, $\mathbf{p}\in\mathcal{P}_p$.

Since $\frac{\partial^{p+1}}{\partial x_i^{p+1}}\mathbf{u}_h=\mathbf{0}$ for $1\leqslant i\leqslant d$, (3.64) yields for $k=p+1$

$$\mathbf{q}_{p+1}(\tau,\boldsymbol{\xi})=\frac{1}{k!}\frac{d^k(\mathbf{u}-\mathbf{u}_h)(T\tau,\boldsymbol{\xi}h,h)}{dh^k}\bigg|_{h=0}=\sum_{|\boldsymbol{\alpha}|\leqslant p+1}\frac{1}{\boldsymbol{\alpha}!}D^{\boldsymbol{\alpha}}(\mathbf{u}-\mathbf{u}_h)(T\tau,\mathbf{0})\boldsymbol{\xi}^{\boldsymbol{\alpha}}$$
$$=\sum_{i=1}^d\frac{1}{(p+1)!}\frac{\partial^{p+1}\mathbf{u}(T\tau,\mathbf{0})}{\partial x_i^{p+1}}\xi_i^{p+1}+\mathbf{p}_1(\tau,\boldsymbol{\xi}), \tag{3.72}$$

where $\mathbf{p}_1(\tau,\boldsymbol{\xi})\in\mathcal{P}_p$. By the definition of $\mathbf{c}_i$ in (3.50c),

$$L_{p+1}(\xi_i)\mathbf{c}_i(T\tau)=\frac{1}{a_{p+1}}\frac{1}{(p+1)!}\frac{\partial^{p+1}\mathbf{u}(T\tau,\mathbf{0})}{\partial x_i^{p+1}}L_{p+1}(\xi_i)$$
$$=\frac{1}{(p+1)!}\frac{\partial^{p+1}\mathbf{u}(T\tau,\mathbf{0})}{\partial x_i^{p+1}}\xi_i^{p+1}+\breve{\mathbf{p}}_i(\tau,\xi_i), \tag{3.73}$$

where $\breve{\mathbf{p}}_i\in\mathcal{P}_p$. Substituting (3.73) into (3.50b) yields

$$\sum_{i=1}^d\hat{\mathbf{r}}_i(\tau,\xi_i)=\sum_{i=1}^d\Big(L_{p+1}(\xi_i)\mathbf{c}_i(T\tau)-L_p(\xi_i)\big(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i(T\tau)+\mathbf{d}_i(T\tau)\big)\Big)$$
$$=\sum_{i=1}^d\frac{1}{(p+1)!}\frac{\partial^{p+1}\mathbf{u}(T\tau,\mathbf{0})}{\partial x_i^{p+1}}\xi_i^{p+1}+\mathbf{p}_2(\tau,\boldsymbol{\xi}), \tag{3.74}$$

where

$$\mathbf{p}_2 = \sum_{i=1}^{d} \big(\check{\mathbf{p}}_i - L_p(\xi_i)\big(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i + \mathbf{d}_i\big)\big) \in \mathcal{P}_p. \tag{3.75}$$

Combining (3.74) and (3.72) yields for $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2 \in \mathcal{P}_p$,

$$\mathbf{q}_{p+1}(\tau, \boldsymbol{\xi}) = \sum_{i=1}^{d} \hat{\mathbf{r}}_i(\tau, \xi_i) + \mathbf{p}(\tau, \boldsymbol{\xi}), \quad \mathbf{p} \in \mathcal{P}_p. \tag{3.76}$$

Substituting (3.76) into (3.71) yields

$$\begin{aligned}
&\sum_{i=1}^{d}\bigg(\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i}\mathbf{A}_i \mathbf{p}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \mathbf{p}\, d\boldsymbol{\sigma}\bigg) \\
&\quad = \sum_{i=1}^{d}\bigg(-\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i}\mathbf{A}_i \sum_{j=1}^{d} \hat{\mathbf{r}}_j\, d\boldsymbol{\xi} + \int_{\Gamma_i} \mathbf{v}^t(\nu_i \mathbf{A}_i)^+ \sum_{j=1}^{d}\hat{\mathbf{r}}_j + \mathbf{v}^t(\nu_i \mathbf{A}_i)^- \sum_{j\in D(i)} \hat{\mathbf{r}}_j\, d\boldsymbol{\sigma}\bigg) \\
&\quad = \sum_{i=1}^{d}\bigg(T_1^i(\mathbf{v}) + \sum_{j\in D(i)} T_2^{i,j}(\mathbf{v}) + T_3^i(\mathbf{v})\bigg), \quad \forall \mathbf{v} \in \mathcal{P}_p,
\end{aligned} \tag{3.77a}$$

where

$$T_1^i(\mathbf{v}) = -\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i}\mathbf{A}_i \hat{\mathbf{r}}_i\, d\boldsymbol{\xi}, \tag{3.77b}$$

$$T_2^{i,j}(\mathbf{v}) = -\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i}\mathbf{A}_i \hat{\mathbf{r}}_j\, d\boldsymbol{\xi} + \int_{\Gamma_i} \mathbf{v}^t \nu_i \mathbf{A}_i \hat{\mathbf{r}}_j\, d\boldsymbol{\sigma}, \tag{3.77c}$$

$$T_3^i(\mathbf{v}) = \int_{\Gamma_i} \mathbf{v}^t(\nu_i \mathbf{A}_i)^+ \hat{\mathbf{r}}_i\, d\boldsymbol{\sigma}, \quad j \in D(i),\ 1 \leqslant i \leqslant d. \tag{3.77d}$$

Now, we show $T_1^i(\mathbf{v}) = T_2^{i,j}(\mathbf{v}) = T_3^i(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathcal{P}_p$, $j \in D(i)$, $1 \leqslant i \leqslant d$.

By the orthogonality properties of Legendre polynomials, we have

$$T_1^i(\mathbf{v}) = -\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i}\mathbf{A}_i \hat{\mathbf{r}}_i\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 1 \leqslant i \leqslant d. \tag{3.78}$$

Integrating (3.77c) by parts w.r.t. ξ_i yields

$$T_2^{i,j}(\mathbf{v}) = \int_{\Delta} \mathbf{v}^t \mathbf{A}_i \frac{\partial \hat{\mathbf{r}}_j}{\partial \xi_i}\, d\boldsymbol{\xi} = 0, \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 1 \leqslant i \leqslant d,\ j \in D(i), \tag{3.79}$$

since $\hat{\mathbf{r}}_j(t, \xi_j)$ is independent of ξ_i for $j \in D(i)$.

Finally, applying $(\nu_i \mathbf{A}_i)^+$ to $\hat{\mathbf{r}}_i$ on Γ_i yields

$$(\nu_i \mathbf{A}_i)^+ \hat{\mathbf{r}}_i\big|_{\Gamma_i^+} = \mathbf{A}_i^+ \hat{\mathbf{r}}_i(\tau, 1), \qquad (\nu_i \mathbf{A}_i)^+ \hat{\mathbf{r}}_i\big|_{\Gamma_i^-} = -\mathbf{A}_i^- \hat{\mathbf{r}}_i(\tau, 0), \quad 1 \leqslant i \leqslant d. \tag{3.80}$$

Using property (2.3g), $L_p(0) = (-1)^p$ and $L_p(1) = 1$, we obtain

$$\mathbf{A}_i^+ \hat{\mathbf{r}}_i(\tau, 1) = \mathbf{A}_i^+ \mathbf{c}_i(T\tau) - \mathbf{A}_i^+ \operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i(T\tau) = \mathbf{0}, \tag{3.81a}$$

$$-\mathbf{A}_i^- \hat{\mathbf{r}}_i(\tau, 0) = -(-1)^{p+1}\mathbf{A}_i^- \mathbf{c}_i(T\tau) + (-1)^p \mathbf{A}_i^- \operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i(T\tau) = \mathbf{0}, \quad 1 \leqslant i \leqslant d. \tag{3.81b}$$

Thus, we have established that $(\nu_i \mathbf{A}_i)^+ \hat{\mathbf{r}}_i|_{\Gamma_i} = \mathbf{0}$, hence

$$T_3^i(\mathbf{v}) = \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \hat{\mathbf{r}}_i\, d\boldsymbol{\sigma} = 0, \quad \forall \mathbf{v} \in \mathcal{P}_p,\ 1 \leqslant i \leqslant d. \tag{3.82}$$

Substituting (3.78), (3.79) and (3.82) into (3.77a) leads to

$$\int_\Delta \frac{\partial \mathbf{v}^t}{\partial \xi_i} \mathbf{A}_i \mathbf{p}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^+ \mathbf{p}\, d\boldsymbol{\sigma} = 0, \quad \mathbf{v} \in \mathcal{P}_p, \tag{3.83}$$

which combined with Lemma 3.5 yields

$$\mathbf{p} \in \bar{\mathcal{P}}_p. \tag{3.84}$$

On the other hand, by (3.66), (3.76) and Lemma 3.4, we obtain

$$\mathbf{p} = \mathbf{q}_{p+1} - \sum_{i=1}^{d} \hat{\mathbf{r}}_i \in \bar{\mathcal{P}}_p^{\perp}. \tag{3.85}$$

Combining (3.84) and (3.85) yields $\mathbf{p} = \mathbf{0}$ on Δ, thus (3.85) leads to

$$\mathbf{q}_{p+1}(\tau, \xi) = \sum_{i=1}^{d} \hat{\mathbf{r}}_i(\tau, \xi_i), \quad \boldsymbol{\xi} \in \Delta. \tag{3.86}$$

Substituting $\mathbf{q}_k = \mathbf{0}$, $0 \leqslant k \leqslant p$, and (3.86) into (3.63) yields (3.50a). This completes the proof. □

Corollary 3.7. *Under the conditions of Theorem 3.6, if all matrices $\mathbf{A}_i$, $1 \leqslant i \leqslant d$, are invertible, then the local discretization error on ω, at $t = \mathcal{O}(1)$ and for $p \geqslant 1$, can be written as*

$$\mathbf{e}(t, h\boldsymbol{\xi}) = h^{p+1} \sum_{i=1}^{d} \big(\mathbf{M}_i R^+_{p+1}(\xi_i) + (\mathbf{I} - \mathbf{M}_i) R^-_{p+1}(\xi_i)\big)\mathbf{c}_i + \mathcal{O}\big(h^{p+2}\big), \tag{3.87}$$

where

$$\mathbf{M}_i = \frac{1}{2}\big(\mathbf{I} + \operatorname{sgn}(\mathbf{A}_i)\big), \quad 1 \leqslant i \leqslant d. \tag{3.88}$$

Moreover, if, for instance, only $\mathbf{A}_1$ is invertible, then the error can be written as

$$\begin{aligned}\mathbf{e}(t, h\boldsymbol{\xi}) &= h^{p+1}\big(\mathbf{M}_1 R^+_{p+1}(\xi_1) + (\mathbf{I} - \mathbf{M}_1) R^-_{p+1}(\xi_1)\big)\mathbf{c}_1 \\ &\quad + h^{p+1} \sum_{i=2}^{d} L_{p+1}(\xi_i)\mathbf{c}_i + L_p(\xi_i)\big(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i + \mathbf{d}_i\big) + \mathcal{O}\big(h^{p+2}\big),\end{aligned} \tag{3.89}$$

where $\mathbf{c}_i$ and $\mathbf{d}_i$ satisfy (3.50c).

The proof of this corollary can be found in [10] for $d = 2$ and is therefore omitted.

In order for the DG solution $\mathbf{u}_h$ to be $\mathcal{O}(h^{p+2})$-superconvergent at few points in element ω, the leading error term shown in Theorem 3.6 has to be zero at these points. This pointwise superconvergence happens only for special hyperbolic problems as shown in the following theorem.

Theorem 3.8. *We let $\bar{\xi}^s_k$, $1 \leqslant k \leqslant p+1$, denote the roots of $R^s_{p+1}(\xi)$, $s = +, -$, shifted to $[0, 1]$. Thus, under the conditions of Theorem 3.6 with $p \geqslant 1$ and $t = \mathcal{O}(1)$,*

(i) *If $\mathbf{z}$ is a unit vector in the union of the spaces $\bigcap_{i=1}^d \mathcal{R}(\mathbf{A}_i^{s_i})$, $s_i = +, -$, then the projection $\mathbf{z}^t\mathbf{e}(t, \mathbf{x})$ of the local error onto span$\{\mathbf{z}\}$ is $\mathcal{O}(h^{p+2})$ superconvergent at the points $(t, h\bar{\boldsymbol{\xi}})$, $\bar{\boldsymbol{\xi}} = (\bar{\xi}^{s_1}_{k_1}, \ldots, \bar{\xi}^{s_d}_{k_d})$, $1 \leqslant k_i \leqslant p+1$, $s_i = +, -$, $1 \leqslant i \leqslant d$, i.e.,*

$$\mathbf{z}^t \mathbf{e}(t, h\bar{\boldsymbol{\xi}}) = \mathcal{O}\big(h^{p+2}\big). \tag{3.90}$$

(ii) *Moreover, if $\gamma_i(a) = \{\mathbf{x} \in (0, h)^d\colon x_i = a\}$, $0 \leqslant a \leqslant h$, and if $\mathbf{v} \in \mathcal{P}_{p-1}$ is a unit vector with respect to the C^∞ norm, then, at $a = h\bar{\xi}^s_k$, we have the superconvergence of the following error averages*

$$\frac{1}{h^{d-1}} \int_{\gamma_i(h\bar{\xi}^s_k)} \mathbf{v}^t \mathbf{A}_i^s \mathbf{e}\, d\mathbf{s} = \mathcal{O}\big(h^{p+2}\big), \quad 1 \leqslant k \leqslant p+1,\ s = +, -,\ 1 \leqslant i \leqslant d, \tag{3.91}$$

and

$$\frac{1}{h^{d-1}} \int_{\gamma_i^s} \mathbf{v}^t \big((\nu_i \mathbf{A}_i)^+ \mathbf{e} + (\nu_i \mathbf{A}_i)^- \mathbf{e}^-\big)\, d\mathbf{s} = \mathcal{O}\big(h^{p+2}\big), \quad s = +, -,\ 1 \leqslant i \leqslant d. \tag{3.92}$$

The proof of this theorem can be found in [10] for $d = 2$ and is therefore omitted.

4. *A posteriori* error estimation

In the following section, we present an efficient *a posteriori* error estimation procedure which consists of solving relatively small local problems. Then we show that the local error estimate is asymptotically exact, *i.e.*, it converges to the discretization error under mesh refinement.

We showed in Theorem 3.6 that the local discretization error for the DG method on a physical element $\omega = (0,h)^d$ can be written as

$$\mathbf{e}(t, h\boldsymbol{\xi}) = h^{p+1}\sum_{i=1}^{d} L_{p+1}(\xi_i)\mathbf{c}_i(t) - L_p(\xi_i)\big(\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i(t) + \mathbf{d}_i(t)\big) + \mathcal{O}\big(h^{p+2}\big), \tag{4.1a}$$

where

$$\mathbf{d}_i(t) \in \mathcal{N}(\mathbf{A}_i) \cap \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k), \quad 1 \leqslant i \leqslant d. \tag{4.1b}$$

Now, we define projections onto the range and the null space of $\mathbf{A}_i$. As shown in [23], $\mathsf{P}_i^{\mathcal{R}} = \mathbf{A}_i\mathbf{A}_i^{\dagger}$ is the orthogonal projection onto $\mathcal{R}(\mathbf{A}_i)$ and $\mathsf{P}_i^{\mathcal{N}} = \mathbf{I} - \mathbf{A}_i^{\dagger}\mathbf{A}_i$ is the orthogonal projection onto $\mathcal{N}(\mathbf{M})$, where $\mathbf{A}_i^{\dagger}$ denotes the pseudoinverse of $\mathbf{A}_i$. Both projections $\mathsf{P}^{\mathcal{R}}$ and $\mathsf{P}^{\mathcal{N}}$ are symmetric.

Since $\mathbf{A}_i$ are symmetric, $\mathcal{R}(\mathbf{A}_i) = \mathcal{N}(\mathbf{A}_i)^{\perp}$ by (2.3a), and we can split $\mathbf{c}_i$ into

$$\mathbf{c}_i = \mathbf{c}_i^{\mathcal{R}} + \mathbf{c}_i^{\mathcal{N}}, \qquad \mathbf{c}_i^{\mathcal{R}} = \mathsf{P}_i^{\mathcal{R}}\mathbf{c}_i \in \mathcal{R}(\mathbf{A}_i), \qquad \mathbf{c}_i^{\mathcal{N}} = \mathsf{P}_i^{\mathcal{N}}\mathbf{c}_i \in \mathcal{N}(\mathbf{A}_i), \quad 1 \leqslant i \leqslant d. \tag{4.2a}$$

By (2.3e), $\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i^{\mathcal{N}} = \mathbf{0}$, which yields $\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i = \operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i^{\mathcal{R}} \in \mathcal{R}(\mathbf{A}_i)$.

Hence, the leading term of the spatial discretization error can be split into two parts as

$$\mathbf{e} = \mathbf{e}^{\mathcal{R}} + \mathbf{e}^{\mathcal{N}} + \mathcal{O}\big(h^{p+2}\big), \tag{4.2b}$$

where

$$\mathbf{e}^{\mathcal{R}}(t, h\boldsymbol{\xi}) = h^{p+1}\sum_{i=1}^{d} L_{p+1}(\xi_i)\mathbf{c}_i^{\mathcal{R}}(t) - L_p(\xi_i)\operatorname{sgn}(\mathbf{A}_i)\mathbf{c}_i^{\mathcal{R}}(t), \tag{4.2c}$$

and

$$\mathbf{e}^{\mathcal{N}}(t, h\boldsymbol{\xi}) = h^{p+1}\sum_{i=1}^{d} L_{p+1}(\xi_i)\mathbf{c}_i^{\mathcal{N}}(t) - L_p(\xi_i)\mathbf{d}_i(t), \quad 1 \leqslant i \leqslant d. \tag{4.2d}$$

We note that for invertible matrices $\mathbf{A}_i$, $1 \leqslant i \leqslant d$, the error component $\mathbf{e}^{\mathcal{N}}(t, \mathbf{x})$ is zero.

Next, we develop an *a posteriori* error estimation procedure for estimating both $\mathbf{e}^{\mathcal{R}}$ and $\mathbf{e}^{\mathcal{N}}$ (if needed). We prove that, for smooth solutions, our local error estimates converge to the true error under mesh refinement. Up to this point we are not able to prove the asymptotic exactness of our global *a posteriori* error estimates. However, computational results for several hyperbolic systems shown in Section 5 suggest that our global *a posteriori* error estimates are asymptotically exact for smooth solutions.

4.1. Stationary component of the error estimate

The *a posteriori* error estimation procedure to compute estimates for $\mathbf{e}^{\mathcal{R}}$ consists of determining

$$\mathbf{E}^{\mathcal{R}}(t, h\boldsymbol{\xi}) = \sum_{j=1}^{d}\big(L_{p+1}(\xi_j)\boldsymbol{\gamma}_j^{\mathcal{R}}(t) - L_p(\xi_j)\operatorname{sgn}(\mathbf{A}_j)\boldsymbol{\gamma}_j^{\mathcal{R}}(t)\big), \quad \boldsymbol{\gamma}_j^{\mathcal{R}} \in \mathcal{R}(\mathbf{A}_j), \tag{4.3a}$$

such that

$$\int_{\omega} L_p\left(\frac{x_i}{h}\right)\mathbf{v}^t\left(\frac{\partial \mathbf{u}_h}{\partial t} + \sum_{j=1}^{d}\mathbf{A}_j\frac{\partial(\mathbf{u}_h + \mathbf{E}^{\mathcal{R}})}{\partial x_j} - \mathbf{g}\right)d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathcal{R}(\mathbf{A}_i),\ 1 \leqslant i \leqslant d. \tag{4.3b}$$

Since $\mathsf{P}_i^{\mathcal{R}}$ is the orthogonal projection onto $\mathcal{R}(\mathbf{A}_i)$, and is symmetric, the columns of $(\mathsf{P}_i^{\mathcal{R}})^t = \mathsf{P}_i^{\mathcal{R}}$ span $\mathcal{R}(\mathbf{A}_i)$. Substituting $\mathbf{v}$ by $(\mathsf{P}_i^{\mathcal{R}})^t$ in (4.3b) yields

$$\int_{\omega} L_p\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{R}}\left(\frac{\partial \mathbf{u}_h}{\partial t}+\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial(\mathbf{u}_h+\mathbf{E}^{\mathcal{R}})}{\partial x_j}-\mathbf{g}\right)d\mathbf{x}=0,\quad 1\leqslant i\leqslant d. \tag{4.4}$$

Substituting (4.3a) into (4.4) and applying the orthogonality properties (2.10), we obtain

$$\mathbf{A}_i\boldsymbol{\gamma}_i^{\mathcal{R}}\int_{\omega} L_p\left(\frac{x_i}{h}\right)L'_{p+1}\left(\frac{x_i}{h}\right)d\mathbf{x}=\mathfrak{r}_{p,i}^{\mathcal{R}}, \tag{4.5a}$$

where $\mathfrak{r}_{p,i}^{\mathcal{R}}$ is the projection of the residual defined as

$$\mathfrak{r}_{p,i}^{\mathcal{R}}=\mathsf{P}_i^{\mathcal{R}}\int_{\omega} L_p\left(\frac{x_i}{h}\right)\left(\mathbf{g}-\frac{\partial \mathbf{u}_h}{\partial t}-\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial \mathbf{u}_h}{\partial x_j}\right)d\mathbf{x},\quad 1\leqslant i\leqslant d. \tag{4.5b}$$

Using (2.10) we further reduce (4.5a), obtaining

$$2h^{d-1}\mathbf{A}_i\boldsymbol{\gamma}_i^{\mathcal{R}}=\mathfrak{r}_{p,i}^{\mathcal{R}},\quad 1\leqslant i\leqslant d. \tag{4.6}$$

Since $\mathfrak{r}_{p,i}^{\mathcal{R}}\in\mathcal{R}(\mathbf{A}_i)$, we can solve (4.6) to find the unique solution $\gamma_i^{\mathcal{R}}\in\mathcal{R}(\mathbf{A}_i)$,

$$\boldsymbol{\gamma}_i^{\mathcal{R}}=\frac{h^{1-d}}{2}\mathbf{A}_i^{\dagger}\mathfrak{r}_{p,i}^{\mathcal{R}},\quad 1\leqslant i\leqslant d. \tag{4.7}$$

Next, we show that this stationary error estimate is asymptotically exact.

Theorem 4.1. *Under the assumptions of Theorem 3.6, let us consider the error estimate*

$$\mathbf{E}^{\mathcal{R}}(t,h\boldsymbol{\xi})=\sum_{i=1}^{d}\big(L_{p+1}(\xi_i)-L_p(\xi_i)\operatorname{sgn}(\mathbf{A}_i)\big)\frac{h^{1-d}}{2}\mathbf{A}_i^{\dagger}\mathfrak{r}_{p,i}^{\mathcal{R}}, \tag{4.8}$$

where $\mathfrak{r}_{p,i}^{\mathcal{R}}$, $1\leqslant i\leqslant d$, are defined in (4.5b).

Then, for $p\geqslant 1$ and $t=\mathcal{O}(1)$,

$$\mathbf{e}^{\mathcal{R}}(t,\mathbf{x})=\mathbf{E}^{\mathcal{R}}(t,\mathbf{x})+\mathcal{O}\big(h^{p+2}\big),\quad \mathbf{x}\in\omega. \tag{4.9}$$

Proof. Since the true solution $\mathbf{u}$ satisfies Eq. (2.1), we have

$$\int_{\omega} L_p\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{R}}\left(\frac{\partial \mathbf{u}}{\partial t}+\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial \mathbf{u}}{\partial x_j}-\mathbf{g}\right)d\mathbf{x}=\mathbf{0},\quad 1\leqslant i\leqslant d. \tag{4.10}$$

Subtracting (4.4) from (4.10) yields

$$\int_{\omega} L_p\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{R}}\left(\frac{\partial \mathbf{e}}{\partial t}+\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial(\mathbf{e}-\mathbf{E}^{\mathcal{R}})}{\partial x_j}\right)d\mathbf{x}=\mathbf{0}. \tag{4.11}$$

Applying $\mathbf{A}_i\mathbf{e}_{,x_i}^{\mathcal{N}}=\mathbf{0}$ and the orthogonality of Legendre polynomials (2.10), (4.11) infers that

$$\int_{\omega} L_p\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{R}}\left(\frac{\partial \mathbf{e}}{\partial t}+\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial(\mathbf{e}^{\mathcal{R}}-\mathbf{E}^{\mathcal{R}})}{\partial x_j}\right)d\mathbf{x}=\mathbf{0}. \tag{4.12}$$

Applying the linear transformations $t=T\tau$ and $\mathbf{x}=h\boldsymbol{\xi}$, (4.12) becomes

$$\int_{\Delta} L_p(\xi_i)\mathsf{P}_i^{\mathcal{R}}\left(\frac{h}{T}\frac{\partial \hat{\mathbf{e}}}{\partial \tau}+\sum_{j=1}^{d}\mathbf{A}_j\frac{\partial(\hat{\mathbf{e}}^{\mathcal{R}}-\hat{\mathbf{E}}^{\mathcal{R}})}{\partial \xi_j}\right)d\boldsymbol{\xi}=\mathbf{0}, \tag{4.13}$$

where $\hat{\mathbf{e}}(\tau,\xi)=\mathbf{e}(\tau T,h\xi)$.

Substituting the definitions of $\mathbf{e}^{\mathcal{R}}$ in (4.2c) and $\mathbf{E}^{\mathcal{R}}$ in (4.3a) into (4.13), noting that $\operatorname{sgn}(\mathbf{A}_i)\frac{\partial}{\partial\tau}\hat{\mathbf{d}}_i=\mathbf{0}$ by (2.3e), and applying the orthogonality properties (2.10), we obtain

$$\mathsf{P}_i^{\mathcal{R}} \int\limits_{\Delta} \frac{h^{p+2}}{T} L_p^2(\xi_i)\operatorname{sgn}(\mathbf{A}_i)\frac{\partial \hat{\mathbf{c}}_i^{\mathcal{R}}}{\partial \tau} + L_p(\xi_i)L'_{p+1}(\xi_i)\mathbf{A}_i\big(h^{p+1}\hat{\mathbf{c}}_i^{\mathcal{R}} - \hat{\boldsymbol{\gamma}}_i^{\mathcal{R}}\big)\,d\boldsymbol{\xi} = \mathcal{O}\big(h^{p+2}\big). \tag{4.14}$$

Using (2.10), and the fact that $\mathsf{P}_i^{\mathcal{R}}\operatorname{sgn}(\mathbf{A}_i) = \operatorname{sgn}(\mathbf{A}_i)$ by (2.3f), (4.14) can be further simplified to

$$\frac{h^{p+2}}{T(2p+1)}\operatorname{sgn}(\mathbf{A}_i)\frac{\partial \hat{\mathbf{c}}^{\mathcal{R}}}{\partial \tau} + 2\mathbf{A}_i\big(h^{p+1}\hat{\mathbf{c}}_i^{\mathcal{R}} - \hat{\boldsymbol{\gamma}}_i^{\mathcal{R}}\big) = \mathcal{O}\big(h^{p+2}\big). \tag{4.15}$$

Thus, at $T = \mathcal{O}(1)$, we have

$$2\mathbf{A}_i\big(h^{p+1}\mathbf{c}_i^{\mathcal{R}} - \gamma_i^{\mathcal{R}}\big) = \mathcal{O}\big(h^{p+2}\big). \tag{4.16}$$

Since $\mathbf{c}_i^{\mathcal{R}}, \boldsymbol{\gamma}_i^{\mathcal{R}} \in \mathcal{R}(\mathbf{A}_i)$, Eq. (4.16) has the unique solution

$$\boldsymbol{\gamma}_i^{\mathcal{R}} = h^{p+1}\mathbf{c}_i^{\mathcal{R}} + \mathcal{O}\big(h^{p+2}\big), \quad 1 \leqslant i \leqslant d. \tag{4.17}$$

This establishes (4.9). □

4.2. Transient component of the error estimate

Here we present a transient *a posteriori* error estimation procedure to compute estimates for $\mathbf{e}^{\mathcal{N}}$. Note that if all $\mathbf{A}_i$, $1 \leqslant i \leqslant d$, are invertible, $\mathbf{e}^{\mathcal{N}} = \mathbf{0}$ by definition in (4.2d).

By Lemma 3.1, the approximations $\pi\mathbf{u}_0$ on ω and $\pi_i^s\mathbf{u}$ on the boundary $\partial\omega$ satisfy

$$\begin{aligned}\mathbf{e}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) - \pi\mathbf{u}_0(\mathbf{x}) \\ &= h^{p+1}\sum_{j=1}^{d} L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j(0) - L_p\left(\frac{x_j}{h}\right)\operatorname{sgn}(\mathbf{A}_j)\mathbf{c}_j(0) + \mathcal{O}\big(h^{p+2}\big), \quad \mathbf{x} \in \omega,\end{aligned} \tag{4.18}$$

$$\begin{aligned}\mathbf{e}^-(t,\mathbf{x}) &= \mathbf{u}(t,\mathbf{x}) - \pi_i^s\mathbf{u}(t,\mathbf{x}) \\ &= h^{p+1}\sum_{j\in D(i)} L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{c}_j(t) - L_p\left(\frac{x_j}{h}\right)\operatorname{sgn}(\mathbf{A}_j)\mathbf{c}_j(t) + \mathcal{O}\big(h^{p+2}\big), \quad \mathbf{x} \in \gamma_i^s,\ s = +,-,\ 1 \leqslant i \leqslant d.\end{aligned} \tag{4.19}$$

We split the error at $t = 0$ into $\mathbf{e} = \mathbf{e}^{\mathcal{R}} + \mathbf{e}^{\mathcal{N}} + \mathcal{O}(h^{p+2})$ as in (4.2b) and define $\mathbf{E}^{\mathcal{N}}(0,\mathbf{x})$ by

$$\mathbf{E}^{\mathcal{N}}(0,\mathbf{x}) = \mathbf{e}^{\mathcal{N}}(0,\mathbf{x}) = h^{p+1}\sum_{i=1}^{d} L_{p+1}\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}}\bar{\mathbf{c}}_i(0), \tag{4.20}$$

where $\mathsf{P}_i^{\mathcal{N}}\bar{\mathbf{c}}_i(0)$ is the projection of $\bar{\mathbf{c}}_i(0)$ into $\mathcal{N}(\mathbf{A}_i)$.

On the boundary, we define $\mathbf{E}^-$ by the leading term of (4.19),

$$\mathbf{E}^-(t,\mathbf{x}) = h^{p+1}\sum_{j\in D(i)} L_{p+1}\left(\frac{x_j}{h}\right)\bar{\mathbf{c}}_{ij}^s(t) - L_p\left(\frac{x_j}{h}\right)\operatorname{sgn}(\mathbf{A}_j)\bar{\mathbf{c}}_{ij}^s(t), \quad \mathbf{x} \in \gamma_i^s,\ 1 \leqslant i \leqslant d. \tag{4.21}$$

Then we define the error estimate for $\mathbf{e}^{\mathcal{N}}$ by determining the coefficients of

$$\mathbf{E}^{\mathcal{N}}(t,\mathbf{x}) = \sum_{j=1}^{d} L_{p+1}\left(\frac{x_j}{h}\right)\boldsymbol{\gamma}_j^{\mathcal{N}}(t) - L_p\left(\frac{x_j}{h}\right)\boldsymbol{\delta}_j(t), \quad \boldsymbol{\gamma}_j^{\mathcal{N}}, \boldsymbol{\delta}_j \in \mathcal{N}(\mathbf{A}_j),\ 1 \leqslant j \leqslant d, \tag{4.22a}$$

such that

$$\begin{aligned}&\int\limits_{\omega} \mathbf{v}^t\left(\frac{\partial(\mathbf{u}_h + \mathbf{E}^{\mathcal{N}})}{\partial t} + \sum_{j=1}^{d}\mathbf{A}_j\frac{\partial \mathbf{u}_h}{\partial x_j} - \mathbf{g}\right)d\mathbf{x} \\ &\quad = \sum_{j=1}^{d}\int\limits_{\gamma_j} \mathbf{v}^t(\nu_j\mathbf{A}_j)^-\big(\mathbf{u}_h + \mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}} - \mathbf{u}_h^- - \mathbf{E}^-\big)\,ds, \quad \forall \mathbf{v} \in \mathcal{E}_p,\end{aligned} \tag{4.22b}$$

where $\mathbf{E}^{\mathcal{R}}$ equals the stationary component defined by (4.3) and

$$\mathcal{E}_p = \left\{ \mathbf{v}(\mathbf{x}) = \sum_{i=1}^{d} \left(L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_i}{h}\right)\mathbf{b}_i \right) : \ \mathbf{a}_i, \mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i) \right\}. \tag{4.22c}$$

The reason for choosing Eq. (4.22b) to estimate $\mathbf{E}^{\mathcal{N}}$ becomes clear when we prove the asymptotic exactness of the $\mathbf{E}^{\mathcal{N}}$ in Theorem 4.3.

Since $\mathsf{P}_i^{\mathcal{N}}$ is the orthogonal projection onto $\mathcal{N}(\mathbf{A}_i)$, and $\mathsf{P}_i^{\mathcal{N}}$ is symmetric, the columns of $L_m(\xi_i)(\mathsf{P}_i^{\mathcal{N}})^t$, $m = p, p+1$, $1 \leqslant i \leqslant d$, span $\mathcal{E}_p$.

Replacing $\mathbf{v}$ in (4.22b) by $L_m(\xi_i)(\mathsf{P}_i^{\mathcal{N}})^t$, $m = p, p+1$, $1 \leqslant i \leqslant d$, yields

$$\begin{aligned} &\int_\omega L_m\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}}\left(\frac{\partial(\mathbf{u}_h + \mathbf{E}^{\mathcal{N}})}{\partial t} + \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{u}_h}{\partial x_j} - \mathbf{g}\right) d\mathbf{x} \\ &\quad = \sum_{j=1}^{d} \int_{\gamma_j} L_m\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}}(\nu_j \mathbf{A}_j)^- \left(\mathbf{u}_h + \mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}} - \mathbf{u}_h^- - \mathbf{E}^-\right) ds, \quad \forall m = p, p+1, \ 1 \leqslant i \leqslant d. \end{aligned} \tag{4.23}$$

By (2.3f), $\mathsf{P}_i^{\mathcal{N}}(\nu_i \mathbf{A}_i)^- = \mathbf{0}$, thus (4.23) can be written as

$$\int_\omega L_m\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}} \frac{\partial \mathbf{E}^{\mathcal{N}}}{\partial t} d\mathbf{x} - \sum_{j \in D(i)} \int_{\gamma_j} L_m\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}}(\nu_j \mathbf{A}_j)^- \mathbf{E}^{\mathcal{N}} d\mathbf{s} = \mathfrak{r}_{m,i}^{\mathcal{N}}, \tag{4.24a}$$

where $\mathfrak{r}_{m,i}^{\mathcal{N}}$ is the projection of the residual given by

$$\begin{aligned} \mathfrak{r}_{m,i}^{\mathcal{N}} &= \mathsf{P}_i^{\mathcal{N}} \int_\omega L_m\left(\frac{x_i}{h}\right)\left(\mathbf{g} - \frac{\partial \mathbf{u}_h}{\partial t} - \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{u}_h}{\partial x_j}\right) d\mathbf{x} \\ &\quad + \mathsf{P}_i^{\mathcal{N}} \sum_{j=1}^{d} \int_{\gamma_j} L_m\left(\frac{x_i}{h}\right)(\nu_j \mathbf{A}_j)^- \left(\mathbf{u}_h + \mathbf{E}^{\mathcal{R}} - \mathbf{u}_h^- - \mathbf{E}^-\right) d\mathbf{s}, \quad m = p, p+1, \ 1 \leqslant i \leqslant d. \end{aligned} \tag{4.24b}$$

For $m = p+1$, we use the orthogonality properties (2.10) to reduce (4.24a) to

$$\int_\omega L_{p+1}^2\left(\frac{x_i}{h}\right)\dot{\boldsymbol{\gamma}}_i^{\mathcal{N}} d\mathbf{x} - \sum_{j \in D(i)} \int_{\gamma_j} L_{p+1}^2\left(\frac{x_i}{h}\right)\mathsf{P}_i^{\mathcal{N}}(\nu_j \mathbf{A}_j)^- \boldsymbol{\gamma}_i^{\mathcal{N}} d\mathbf{s} = \mathfrak{r}_{p+1,i}^{\mathcal{N}}, \tag{4.25}$$

which by (2.10) is equal to

$$\dot{\boldsymbol{\gamma}}_i^{\mathcal{N}} = \frac{1}{h}\mathsf{P}_i^{\mathcal{N}} \sum_{j \in D(i)} (\mathbf{A}_j^- - \mathbf{A}_j^+)\boldsymbol{\gamma}_i^{\mathcal{N}} + \frac{2p+3}{h^d}\mathfrak{r}_{p+1,i}^{\mathcal{N}}. \tag{4.26a}$$

For $m = p$, we get similarly

$$\dot{\boldsymbol{\delta}}_i = \frac{1}{h}\mathsf{P}_i^{\mathcal{N}} \sum_{j \in D(i)} (\mathbf{A}_j^- - \mathbf{A}_j^+)\boldsymbol{\delta}_i + \frac{2p+1}{h^d}\mathfrak{r}_{p,i}^{\mathcal{N}}, \tag{4.26b}$$

subject to the initial conditions

$$\boldsymbol{\gamma}_i^{\mathcal{N}}(0) = h^{p+1}\mathsf{P}_i^{\mathcal{N}}\bar{\mathbf{c}}_i(0), \qquad \boldsymbol{\delta}_i(0) = \mathbf{0}. \tag{4.26c}$$

Note that (4.26) and (4.24b) ensures that $\boldsymbol{\gamma}_i^{\mathcal{N}}, \boldsymbol{\delta}_i \in \mathcal{N}(\mathbf{A}_i)$, $1 \leqslant i \leqslant d$.

Then (4.26) and (4.24b) together describe the procedure to obtain the coefficients of $\mathbf{E}^{\mathcal{N}}$.

4.3. Asymptotic exactness of the transient component of the error estimate

In this subsection, we show the asymptotic exactness of the error estimate.

First, we show some properties of the subspace $\bar{\mathcal{E}}_p \subseteq \mathcal{E}_p$, defined by

$$\bar{\mathcal{E}}_p = \left\{ \mathbf{v}(\mathbf{x}) = \sum_{i=1}^{d} \left(L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_i}{h}\right)\mathbf{b}_i \right) : \ \mathbf{a}_i, \mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i) \cap \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k) \right\}. \tag{4.27a}$$

We can split $\mathcal{E}_p = \bar{\mathcal{E}}_p \oplus \bar{\mathcal{E}}_p^{\perp}$, where the orthogonal complement of $\bar{\mathcal{E}}_p$ in $\mathcal{E}_p$ is defined by

$$\bar{\mathcal{E}}_p^{\perp} = \left\{ \mathbf{v}(\mathbf{x}) \in \mathcal{E}_p \colon \int_{\omega} \mathbf{w}^t \mathbf{v}\, d\mathbf{x} = 0,\ \forall \mathbf{w} \in \bar{\mathcal{E}}_p \right\} \tag{4.27b}$$

$$= \left\{ \mathbf{v}(\mathbf{x}) = \sum_{i=1}^{d} \left(L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_i}{h}\right)\mathbf{b}_i \right) \colon \mathbf{a}_i, \mathbf{b}_i \in \bigcap_{k=1}^{d} \mathcal{N}(\mathbf{A}_k) \right\}. \tag{4.27c}$$

Here we used the fact that

$$\bigcap_{k=1}^{d} \mathcal{N}(\mathbf{A}_k) = \left(\bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k) \right)^{\perp}. \tag{4.28}$$

Note that, if $\bigcap_{k=1}^{d} \mathcal{N}(\mathbf{A}_k) = \{\mathbf{0}\}$, then $\bar{\mathcal{E}}_p = \mathcal{E}_p$ and $\bar{\mathcal{E}}_p^{\perp} = \{\mathbf{0}\}$.

Lemma 4.2. *If $\mathbf{q} \in \bar{\mathcal{E}}_p$ satisfies the orthogonality condition*

$$\sum_{i=1}^{d} \left(\int_{\Delta} \mathbf{v}^t \mathbf{A}_i \frac{\partial \mathbf{q}}{\partial \xi_i}\, d\boldsymbol{\xi} - \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^{-} \mathbf{q}\, d\boldsymbol{\sigma} \right) = 0, \quad \forall \mathbf{v} \in \mathcal{E}_p, \tag{4.29}$$

then $\mathbf{q} = \mathbf{0}$.

Proof. First, we integrate Eq. (4.29) by parts to write

$$\sum_{i=1}^{d} \left(-\int_{\Delta} \frac{\partial \mathbf{v}^t}{\partial \xi_i} \mathbf{A}_i \mathbf{q}\, d\boldsymbol{\xi} + \int_{\Gamma_i} \mathbf{v}^t (\nu_i \mathbf{A}_i)^{+} \mathbf{q}\, d\boldsymbol{\sigma} \right) = 0, \quad \forall \mathbf{v} \in \mathcal{E}_p. \tag{4.30}$$

Adding (4.29) and (4.30) and setting $\mathbf{v} = \mathbf{q}$, the integral on Δ vanishes because of the symmetry of $\mathbf{A}_i$, $1 \leqslant i \leqslant d$, and we get

$$\sum_{i=1}^{d} \int_{\Gamma_i} \mathbf{q}^t \left(\mathbf{A}_i^{+} - \mathbf{A}_i^{-}\right) \mathbf{q}\, d\boldsymbol{\sigma} = 0. \tag{4.31}$$

Since $(\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$ is positive semi-definite by Lemma 2.1, there exists a matrix $\mathbf{L}_i$ such that $\mathbf{L}_i^t \mathbf{L}_i = (\mathbf{A}_i^{+} - \mathbf{A}_i^{-})$, and (4.31) yields

$$\sum_{i=1}^{d} \int_{\Gamma_i} \|\mathbf{L}_i \mathbf{q}\|^2\, d\boldsymbol{\sigma} = 0, \tag{4.32}$$

which yields

$$\mathbf{L}_i \mathbf{q}|_{\Gamma_i} = \mathbf{0}, \tag{4.33}$$

and therefore

$$\left(\mathbf{A}_i^{+} - \mathbf{A}_i^{-}\right)\mathbf{q}\big|_{\Gamma_i} = \mathbf{L}_i^t \mathbf{L}_i \mathbf{q}\big|_{\Gamma_i} = \mathbf{0}, \quad 1 \leqslant i \leqslant d. \tag{4.34}$$

By property (2.3e) and (4.34) we obtain

$$\mathbf{A}_i \mathbf{q}|_{\Gamma_i} = \mathbf{0}, \quad 1 \leqslant i \leqslant d. \tag{4.35}$$

Since $\mathbf{q} \in \bar{\mathcal{E}}_p$,

$$\mathbf{q}(\boldsymbol{\xi}) = \sum_{j=1}^{d} L_{p+1}(\xi_j)\mathbf{a}_j - L_p(\xi_j)\mathbf{b}_j, \quad \mathbf{a}_j, \mathbf{b}_j \in \mathcal{N}(\mathbf{A}_j) \cap \bigoplus_{k=1}^{d} \mathcal{R}(\mathbf{A}_k). \tag{4.36}$$

Substituting (4.36) into (4.35) yields

$$\sum_{j=1}^{d} L_{p+1}(\xi_j)(\mathbf{A}_i \mathbf{a}_j) - L_p(\xi_j)(\mathbf{A}_i \mathbf{b}_j) = \mathbf{0}, \quad \boldsymbol{\xi} \in \Gamma_i,\ 1 \leqslant i \leqslant d. \tag{4.37}$$

Since $\mathbf{a}_j, \mathbf{b}_j \in \mathcal{N}(\mathbf{A}_j)$ and $L_p(\xi_j)$, $L_{p+1}(\xi_j)$, $j \in D(i)$ are pairwise orthogonal functions on Γ_i and therefore linearly independent, (4.37) yields

$$\mathbf{A}_i\mathbf{a}_j = \mathbf{0}, \qquad \mathbf{A}_i\mathbf{b}_j = \mathbf{0}, \quad 1 \leqslant i, j \leqslant d. \tag{4.38}$$

Thus, $\mathbf{a}_j, \mathbf{b}_j \in \bigcup_{i=1}^d \mathcal{N}(\mathbf{A}_i)$, which, when combined with $\mathbf{a}_j, \mathbf{b}_j \in \bigoplus_{i=1}^d \mathcal{R}(\mathbf{A}_k)$ and (4.28), yields

$$\mathbf{a}_j = \mathbf{b}_j = \mathbf{0}, \quad 1 \leqslant j \leqslant d, \tag{4.39}$$

or equivalently $\mathbf{q} = \mathbf{0}$. $\square$

Theorem 4.3. *Under the assumptions of Theorem 3.6, assume further that $\mathbf{u}_h$ is computed by approximating the initial conditions by $\pi\mathbf{u}_0$ and let $\mathbf{E}^{\mathcal{R}}$ be computed as specified in Theorem 4.1. Consider the estimate*

$$\mathbf{E}^{\mathcal{N}}(t, h\boldsymbol{\xi}) = \sum_{j=1}^{d} \big(L_{p+1}(\xi_j)\boldsymbol{\gamma}_j^{\mathcal{N}}(t) - L_p(\xi_j)\boldsymbol{\delta}_j(t)\big), \tag{4.40}$$

where $\boldsymbol{\gamma}_i^{\mathcal{N}}$, $\boldsymbol{\delta}_i$, $1 \leqslant i \leqslant d$, are solutions of (4.26) and (4.24b).

Then, at $t = \mathcal{O}(1)$ and for $p \geqslant 1$,

$$\mathbf{e}^{\mathcal{N}}(t, \mathbf{x}) = \mathbf{E}^{\mathcal{N}}(t, \mathbf{x}) + \mathcal{O}\big(h^{p+2}\big), \quad \mathbf{x} \in \omega. \tag{4.41}$$

Proof. First, we derive a *orthogonality condition* for the remainder $\mathbf{e} - \mathbf{E}^{\mathcal{R}} - \mathbf{E}^{\mathcal{N}}$.

Since the true solution $\mathbf{u}$ is continuous and $\mathbf{u} = \mathbf{u}^-$ on $\partial\omega$, $\mathbf{u}$ satisfies

$$\int_\omega \mathbf{v}^t \left(\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{u}}{\partial x_j} - \mathbf{g}\right) d\mathbf{x} = \sum_{j=1}^{d} \int_{\gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- \big(\mathbf{u} - \mathbf{u}^-\big)\, d\mathbf{s}, \quad \forall \mathbf{v} \in \mathcal{E}_p. \tag{4.42}$$

Subtracting (4.22b) from (4.42) gives

$$\int_\omega \mathbf{v}^t \left(\frac{\partial(\mathbf{e} - \mathbf{E}^{\mathcal{N}})}{\partial t} + \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{e}}{\partial x_j}\right) d\mathbf{x} = \sum_{j=1}^{d} \int_{\gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- \big(\mathbf{e} - \mathbf{E}^{\mathcal{R}} - \mathbf{E}^{\mathcal{N}} - \mathbf{e}^- + \mathbf{E}^-\big)\, d\mathbf{s}, \quad \forall \mathbf{v} \in \mathcal{E}_p. \tag{4.43}$$

Since $\mathbf{v} \in \mathcal{E}_p$, we can write

$$\mathbf{v}(\mathbf{x}) = \sum_{i=1}^{d} L_{p+1}\left(\frac{x_i}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_i}{h}\right)\mathbf{b}_i, \quad \mathbf{a}_i, \mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i), \tag{4.44}$$

while $\mathbf{E}^{\mathcal{R}}$ is defined in (4.3a) as

$$\mathbf{E}^{\mathcal{R}}(t, h\boldsymbol{\xi}) = \sum_{j=1}^{d} L_{p+1}(\xi_j)\boldsymbol{\gamma}_j^{\mathcal{R}}(t) - L_p(\xi_j)\operatorname{sgn}(\mathbf{A}_j)\boldsymbol{\gamma}_j^{\mathcal{R}}(t), \quad \boldsymbol{\gamma}_j^{\mathcal{R}} \in \mathcal{N}(\mathbf{A}_j)^{\perp}. \tag{4.45}$$

By (2.3f) and (2.3a), $\operatorname{sgn}(\mathbf{A}_j)\dot{\boldsymbol{\gamma}}_j^{\mathcal{R}} \in \mathcal{R}(\mathbf{A}_j) = \mathcal{N}(\mathbf{A}_j)^{\perp}$, which, together with (4.45), yields

$$\big\langle \mathbf{a}_i, \dot{\gamma}_i^{\mathcal{R}}(t)\big\rangle = 0, \qquad \big\langle \mathbf{b}_i, \operatorname{sgn}(\mathbf{A}_i)\dot{\gamma}_i^{\mathcal{R}}(t)\big\rangle = 0, \quad 1 \leqslant i \leqslant d. \tag{4.46}$$

By substituting $\mathbf{v}$ and $\mathbf{E}^{\mathcal{R}}$, as defined in (4.44) and (4.45), into $\int_\omega \mathbf{v}^t \frac{\partial \mathbf{E}^{\mathcal{R}}}{\partial t}\, d\mathbf{x}$ and applying the orthogonality property (2.10), we obtain

$$\begin{aligned}
\int_\omega \mathbf{v}^t \frac{\partial \mathbf{E}^{\mathcal{R}}}{\partial t}\, d\mathbf{x} &= \sum_{i=1}^{d}\sum_{j=1}^{d} \int_\omega \left(L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{a}_j - L_p\left(\frac{x_j}{h}\right)\mathbf{b}_j\right)^t \left(L_{p+1}\left(\frac{x_i}{h}\right)\dot{\boldsymbol{\gamma}}_i^{\mathcal{R}}(t) - L_p\left(\frac{x_i}{h}\right)\operatorname{sgn}(\mathbf{A}_i)\dot{\boldsymbol{\gamma}}_i^{\mathcal{R}}(t)\right) d\mathbf{x} \\
&= \sum_{i=1}^{d} \int_\omega L_{p+1}^2\left(\frac{x_i}{h}\right)\big\langle \mathbf{a}_i, \dot{\boldsymbol{\gamma}}_i^{\mathcal{R}}(t)\big\rangle + L_p^2\left(\frac{x_i}{h}\right)\big\langle \mathbf{b}_i, \operatorname{sgn}(\mathbf{A}_i)\dot{\boldsymbol{\gamma}}_i^{\mathcal{R}}(t)\big\rangle\, d\mathbf{x} \\
&= 0, \quad \forall \mathbf{v} \in \mathcal{E}_p.
\end{aligned} \tag{4.47}$$

Furthermore, by substituting $\mathbf{v}$, $\mathbf{E}^{\mathcal{R}}$ and $\mathbf{E}^{\mathcal{N}}$, as defined in (4.44), (4.45) and (4.22a), into $\int_\omega \mathbf{v}^t \mathbf{A}_i \frac{\partial(\mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}})}{\partial x_i}\, d\mathbf{x}$ and applying the orthogonality property (2.10), we obtain

$$\int_\omega \mathbf{v}^t \mathbf{A}_i \frac{\partial(\mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}})}{\partial x_i} d\mathbf{x} = \frac{1}{h}\sum_{j=1}^{d}\int_\omega \left(L_{p+1}\left(\frac{x_j}{h}\right)\mathbf{a}_i - L_p\left(\frac{x_j}{h}\right)\mathbf{b}_i\right)^t \mathbf{A}_i$$
$$\times \left(L'_{p+1}\left(\frac{x_i}{h}\right)\left(\boldsymbol{\gamma}_i^{\mathcal{R}} + \boldsymbol{\gamma}_i^{\mathcal{N}}\right) - L'_p\left(\frac{x_i}{h}\right)\left(\operatorname{sgn}(\mathbf{A}_i)\boldsymbol{\gamma}_i^{\mathcal{R}} + \delta_i\right)\right) d\mathbf{x}$$
$$= \frac{1}{h}\int_\omega L_p\left(\frac{x_i}{h}\right)L'_{p+1}\left(\frac{x_i}{h}\right)(\mathbf{A}_i\mathbf{b}_i)^t\left(\boldsymbol{\gamma}_i^{\mathcal{R}} + \boldsymbol{\gamma}_i^{\mathcal{N}}\right) d\mathbf{x}$$
$$= 0, \quad \forall \mathbf{v} \in \mathcal{E}_p,\ 1 \leqslant i \leqslant d, \tag{4.48}$$

where we used the fact that $\mathbf{b}_i \in \mathcal{N}(\mathbf{A}_i)$.

Subtracting (4.47) and (4.48) from (4.43) yields an orthogonality condition for $\boldsymbol{\epsilon} = \mathbf{e} - \mathbf{E}^{\mathcal{R}} - \mathbf{E}^{\mathcal{N}}$ and $\epsilon^- = \mathbf{e}^- - \mathbf{E}^-$

$$\int_\omega \mathbf{v}^t\left(\frac{\partial \epsilon}{\partial t} + \sum_{j=1}^{d}\mathbf{A}_j\frac{\partial \epsilon}{\partial x_j}\right) d\mathbf{x} = \sum_{j=1}^{d}\int_{\gamma_j}\mathbf{v}^t(\nu_j\mathbf{A}_j)^-\left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^-\right) d\mathbf{s}, \quad \forall \mathbf{v} \in \mathcal{E}_p. \tag{4.49}$$

By (4.2b) we can write

$$\boldsymbol{\epsilon} = \left(\mathbf{e}^{\mathcal{R}} - \mathbf{E}^{\mathcal{R}}\right) + \left(\mathbf{e}^{\mathcal{N}} - \mathbf{E}^{\mathcal{N}}\right) + \mathcal{O}\left(h^{p+2}\right), \tag{4.50}$$

which, since $\mathbf{e}^{\mathcal{R}} - \mathbf{E}^{\mathcal{R}} = \mathcal{O}(h^{p+2})$ by Theorem 4.1, infers

$$\boldsymbol{\epsilon} = \left(\mathbf{e}^{\mathcal{N}} - \mathbf{E}^{\mathcal{N}}\right) + \mathcal{O}\left(h^{p+2}\right). \tag{4.51}$$

In the remainder of the proof we show $\boldsymbol{\epsilon} = \mathcal{O}(h^{p+2})$.

Since, by definition, $(\mathbf{e}^{\mathcal{N}} - \mathbf{E}^{\mathcal{N}}) \in \mathcal{E}_p = \bar{\mathcal{E}}_p \oplus \bar{\mathcal{E}}_p^{\perp}$, we can split $\boldsymbol{\epsilon}$ into

$$\boldsymbol{\epsilon} = \bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\epsilon}}^{\perp} + \mathcal{O}\left(h^{p+2}\right), \quad \bar{\boldsymbol{\epsilon}} \in \bar{\mathcal{E}}_p,\ \bar{\boldsymbol{\epsilon}}^{\perp} \in \bar{\mathcal{E}}_p^{\perp}, \tag{4.52}$$

where $\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\epsilon}}^{\perp}$ are the projections of $(\mathbf{e}^{\mathcal{N}} - \mathbf{E}^{\mathcal{N}})$ into $\bar{\mathcal{E}}_p$ and $\bar{\mathcal{E}}_p^{\perp}$.

First, we show that $\bar{\boldsymbol{\epsilon}}^{\perp} = \mathcal{O}(h^{p+2})$.

By property (2.3e) and the fact that $\frac{\partial}{\partial \tau}\bar{\boldsymbol{\epsilon}}^{\perp} \in \bar{\mathcal{E}}_p^{\perp}$, we have $\mathbf{A}\frac{\partial}{\partial \tau}\bar{\boldsymbol{\epsilon}}^{\perp} = \mathbf{A}_i^s\frac{\partial}{\partial \tau}\bar{\boldsymbol{\epsilon}}^{\perp} = \mathbf{0}$, $s = +, -$, $1 \leqslant i \leqslant d$. Thus, substituting $\frac{\partial}{\partial \tau}\bar{\boldsymbol{\epsilon}}^{\perp}$ for $\mathbf{v}$ in (4.49), the right side vanishes, and we obtain

$$\int_\omega \left(\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right)^t \frac{\partial \boldsymbol{\epsilon}}{\partial t} d\mathbf{x} = \int_\omega \left(\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right)^t \left(\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t} + \mathcal{O}\left(h^{p+2}\right)\right) d\mathbf{x} = 0, \tag{4.53}$$

which, by applying the Cauchy–Schwarz inequality, infers

$$\left\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right\|_{2,\omega}^2 = -\int_\omega \left(\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right)^t \mathcal{O}\left(h^{p+2}\right) d\mathbf{x} \leqslant C h^{p+2}|\omega|^{1/2}\left\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right\|_{2,\omega}. \tag{4.54}$$

Dividing (4.54) by $\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\|_{2,\omega}$ yields

$$\left\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right\|_{2,\omega} \leqslant C h^{p+2}|\omega|^{1/2}. \tag{4.55}$$

Applying inverse inequality (3.3) to (4.55), we obtain

$$\left\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right\|_{\infty,\omega} \leqslant C_1|\omega|^{-1/2}\left\|\frac{\partial \bar{\boldsymbol{\epsilon}}^{\perp}}{\partial t}\right\|_{2,\omega} \leqslant C h^{p+2}. \tag{4.56}$$

By initial conditions (4.20), $\mathbf{E}^{\mathcal{N}}(0,\mathbf{x}) = \mathbf{e}^{\mathcal{N}}(0,\mathbf{x})$, $\mathbf{x} \in \omega$, thus

$$\left(\bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\epsilon}}^{\perp}\right)(0,\mathbf{x}) = \left(\mathbf{e}^{\mathcal{N}} - \mathbf{E}^{\mathcal{N}}\right)(0,\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \omega. \tag{4.57}$$

Thus, $\bar{\boldsymbol{\epsilon}}^{\perp}(0,\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \omega$, which together with (4.56) and the Fundamental Theorem of Calculus yields $\bar{\boldsymbol{\epsilon}}^{\perp} = \mathcal{O}(h^{p+2})$, and therefore

$$\boldsymbol{\epsilon} = \bar{\boldsymbol{\epsilon}} + \mathcal{O}\left(h^{p+2}\right), \quad \bar{\boldsymbol{\epsilon}} \in \bar{\mathcal{E}}_p. \tag{4.58}$$

Applying the linear transformations $t = T\tau$, $T > 0$, and $\mathbf{x} = h\boldsymbol{\xi}$, (4.49) becomes

$$\int_{\Delta} \mathbf{v}^t \left(\frac{h}{T} \frac{\partial \hat{\boldsymbol{\epsilon}}}{\partial \tau} + \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \hat{\boldsymbol{\epsilon}}}{\partial \xi_j} \right) d\boldsymbol{\xi} = \sum_{j=1}^{d} \int_{\Gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- (\hat{\boldsymbol{\epsilon}} - \hat{\boldsymbol{\epsilon}}^-)\, d\boldsymbol{\sigma}, \quad \forall \mathbf{v} \in \mathcal{E}_p, \tag{4.59}$$

where $\hat{\boldsymbol{\epsilon}}(\tau, \boldsymbol{\xi}) = \boldsymbol{\epsilon}(T\tau, h\boldsymbol{\xi})$.

The Maclaurin series of $\bar{\boldsymbol{\epsilon}} \in \bar{\mathcal{E}}_p$ with respect to h is

$$\bar{\boldsymbol{\epsilon}}(t, h\boldsymbol{\xi}) = \sum_{k=0}^{\infty} h^k \mathbf{q}_k(t, \boldsymbol{\xi}), \quad \mathbf{q}_k \in \bar{\mathcal{E}}_p,\ k \geqslant 0, \tag{4.60}$$

which together with (4.58) yields

$$\hat{\boldsymbol{\epsilon}}(t, \boldsymbol{\xi}) = \sum_{k=0}^{p+1} h^k \mathbf{q}_k(t, \boldsymbol{\xi}) + \mathcal{O}\big(h^{p+2}\big). \tag{4.61}$$

By (4.19) and (4.21),

$$\hat{\boldsymbol{\epsilon}}^-(t, \boldsymbol{\xi}) = \mathcal{O}\big(h^{p+2}\big). \tag{4.62}$$

Substituting (4.61) and (4.62) into (4.49) yields

$$\sum_{k=0}^{p+1} h^k \left(\int_{\Delta} \mathbf{v}^t \left(\frac{h}{T} \frac{\partial \mathbf{q}_k}{\partial \tau} + \sum_{j=1}^{d} \mathbf{A}_j \frac{\partial \mathbf{q}_k}{\partial \xi_j} \right) d\boldsymbol{\xi} - \sum_{j=1}^{d} \int_{\Gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- \mathbf{q}_k\, d\boldsymbol{\sigma} \right) = \mathcal{O}\big(h^{p+2}\big), \quad \forall \mathbf{v} \in \mathcal{E}_p, \tag{4.63}$$

which infers that all terms of the same power in h are zero.

The $\mathcal{O}(1)$ term leads to the orthogonality condition for $\mathbf{q}_0$,

$$\sum_{j=1}^{d} \left(\int_{\Delta} \mathbf{v}^t \mathbf{A}_j \frac{\partial \mathbf{q}_0}{\partial \xi_j}\, d\boldsymbol{\xi} - \int_{\Gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- \mathbf{q}_0\, d\boldsymbol{\sigma} \right) = 0, \quad \forall \mathbf{v} \in \mathcal{E}_p. \tag{4.64}$$

Since $\mathbf{q}_0 \in \bar{\mathcal{E}}_p$ by (4.60) and satisfies (4.64), Lemma 4.2 infers $\mathbf{q}_0 = \mathbf{0}$.

Using induction, we assume that $\mathbf{q}_l = \mathbf{0}$, $0 \leqslant l \leqslant k-1$, $k \leqslant p+1$, and apply the $\mathcal{O}(h^k)$ term to obtain the orthogonality condition

$$\sum_{j=1}^{d} \left(\int_{\Delta} \mathbf{v}^t \mathbf{A}_j \frac{\partial \mathbf{q}_k}{\partial \xi_j}\, d\boldsymbol{\xi} - \int_{\Gamma_j} \mathbf{v}^t (\nu_j \mathbf{A}_j)^- \mathbf{q}_k\, d\boldsymbol{\sigma} \right) = 0, \quad \forall \mathbf{v} \in \mathcal{E}_p, \tag{4.65}$$

which, by Lemma 4.2 and (4.60), infers $\mathbf{q}_k = \mathbf{0}$, $k \leqslant p+1$.

Substituting $\mathbf{q}_k = \mathbf{0}$, $k \leqslant p+1$ into (4.61) yields $\boldsymbol{\epsilon} = \mathcal{O}(h^{p+2})$, which, when substituted into (4.51), yields (4.41). This completes the proof. □

5. Computational examples

In Section 4 we established the asymptotic exactness of our *a posteriori* error estimates for a local DG formulation. Here, we present computational results for several hyperbolic systems that suggest that our *a posteriori* error estimates are globally asymptotically exact for smooth solutions. Since in [10] we showed computational results for linear hyperbolic systems that satisfy the assumptions of Lemma 3.2 in [10], here we show computational results for linear hyperbolic systems which violate those assumptions and use $\pi \mathbf{u}_0$ to approximate the initial conditions.

The accuracy of *a posteriori* error estimates is measured by the global effectivity index with respect to the L^2 norm

$$\theta = \frac{\|\mathbf{E}\|_{2,\Omega}}{\|\mathbf{e}\|_{2,\Omega}}, \tag{5.1}$$

where $\mathbf{E}$ is either $\mathbf{E}^{\mathcal{R}}$ or $\mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}}$.

We also need the componentwise L^2-error,

$$\|\mathbf{e}\|^* = \big(\|e_1\|_{2,\Omega}, \ldots, \|e_m\|_{2,\Omega}\big)^t, \tag{5.2}$$

and the componentwise effectivity index,

$$\theta^* = \left(\frac{\|E_1\|_{2,\Omega}}{\|e_1\|_{2,\Omega}}, \ldots, \frac{\|E_m\|_{2,\Omega}}{\|e_m\|_{2,\Omega}} \right)^t. \tag{5.3}$$

where $\mathbf{E} = (E_1, \ldots, E_m)^t$ and $\mathbf{e} = (e_1, \ldots, e_m)^t$.

Ideally, the effectivity indices should approach unity under mesh refinement.

Example 5.1. Let us consider Maxwell's equations of electromagnetism,

$$\varepsilon_0 \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} = \nabla \times \boldsymbol{\mathcal{H}}, \qquad \mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} = \nabla \times \boldsymbol{\mathcal{E}}, \tag{5.4a}$$

$$\nabla \cdot \boldsymbol{\mathcal{E}} = 0, \qquad \nabla \cdot \boldsymbol{\mathcal{H}} = 0, \tag{5.4b}$$

where $\boldsymbol{\mathcal{E}}(t, \mathbf{x}) = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)^t$ and $\boldsymbol{\mathcal{H}}(t, \mathbf{x}) = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)^t$ denote the *electric and magnetic field* and $\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{N\,A^{-2}}$ and $\varepsilon_0 = c_0^{-2}\mu_0^{-1}$ denote the *magnetic and electric permittivity in free space*, respectively, with $c_0 = 299{,}792{,}458\ \mathrm{m\,s^{-2}}$ being the *speed of light*. Eqs. (5.4b) are satisfied for all t, if the initial conditions satisfy them.

For a transverse electric wave traveling in the x_1x_2-plane, $\mathcal{E}_3 = \mathcal{H}_1 = \mathcal{H}_2 = 0$. If we choose space and time units such that $c_0 = 1$, (5.4a) yields the linear symmetric hyperbolic system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial x_1} + \mathbf{A}_2 \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{g}, \quad \mathbf{x} \in \Omega = (0,1)^2,\ 0 < t < 1, \tag{5.5a}$$

where

$$\mathbf{u} = \begin{pmatrix} \sqrt{\varepsilon_0}\mathcal{E}_1 \\ \sqrt{\varepsilon_0}\mathcal{E}_2 \\ \sqrt{\mu_0}\mathcal{H}_3 \end{pmatrix}, \qquad \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{5.5b}$$

We select initial and boundary conditions such that the true solution is

$$\mathbf{u}(t, \mathbf{x}) = (1, -1, \sqrt{2})^t \exp\left(t + \frac{x+y}{\sqrt{2}}\right), \quad \forall 0 \leqslant t \leqslant 1,\ \mathbf{x} \in \Omega. \tag{5.5c}$$

Both matrices $\mathbf{A}_1, \mathbf{A}_2$ are singular and admit the eigenvalues $\{-1, 0, 1\}$. Moreover, the eigenvectors $(1,0,0)^t$ and $(0,1,0)^t$ are associated with the zero eigenvalue for $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Applying our theory, the stationary error estimate $\mathbf{E}^{\mathcal{R}}$ can only accurately approximate the component of the error lying in $(\mathcal{N}(\mathbf{A}_1) \bigoplus \mathcal{N}(\mathbf{A}_2))^\perp = \operatorname{span}\{(0,0,1)^t\}$, *i.e.*, only $E_3^{\mathcal{R}}$ is an accurate estimate of e_3.

To validate our theory, we solve (5.5) on uniform meshes having $N = 10^2, 20^2, 30^2, 40^2$ elements for $p = 1, 2, 3$. We present the componentwise L^2-errors and effectivity indices corresponding to the stationary error estimate $\mathbf{E}^{\mathcal{R}}$ at $t = 1$ in Table 1. We observe that the stationary error estimates are accurate in the third component only which is in full agreement with Theorem 4.1. In Table 2, we present the L^2-errors and effectivity indices for the transient error estimate $\mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}}$ at $t = 1$. We observe that global effectivity indices for the transient error estimate converge to unity under mesh refinement, while and the effectivity indices for the stationary estimate converge to unity in the third component. Furthermore, we plot the effectivity indices for the transient error estimate versus time in Fig. 2 to show that $\mathbf{E}^{\mathcal{R}} + \mathbf{E}^{\mathcal{N}}$ is asymptotically exact at all times, which is in full agreement with Theorem 4.3. Thus, the transient effectivity indices stay close to unity.

Example 5.2. For a general electromagnetic wave, if we choose space and time units such that $c_0 = 1$, we can write (5.4a) as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}_1 \frac{\partial \mathbf{u}}{\partial x_1} + \mathbf{A}_2 \frac{\partial \mathbf{u}}{\partial x_2} + \mathbf{A}_3 \frac{\partial \mathbf{u}}{\partial x_3} = \mathbf{0}, \quad \mathbf{x} \in \Omega = (0,1)^3,\ 0 < t < 1, \tag{5.6a}$$

where

$$\mathbf{u} = \begin{pmatrix} \sqrt{\varepsilon_0}\mathcal{E}_1 \\ \sqrt{\varepsilon_0}\mathcal{E}_2 \\ \sqrt{\varepsilon_0}\mathcal{E}_3 \\ \sqrt{\mu_0}\mathcal{H}_1 \\ \sqrt{\mu_0}\mathcal{H}_2 \\ \sqrt{\mu_0}\mathcal{H}_3 \end{pmatrix}, \qquad \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.6b}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.6c}$$

Table 1
Componentwise $L^2(\Omega)$-errors $\|\mathbf{e}\|^*$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}\|^*$ and their order of convergence. Global effectivity indices corresponding to stationary estimates for Example 5.1 at $t=1$.

p	N	$\|\mathbf{e}\|^*$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}\|^*$	Order	θ^*
1	20^2	6.5522e−4 6.5522e−4 7.4984e−4	–	3.1339e−4 3.1339e−4 1.6742e−5	–	0.8869 0.8869 1.0098
	30^2	2.9219e−4 2.9219e−4 3.3401e−4	1.9917 1.9917 1.9944	1.3995e−4 1.3995e−4 5.1471e−6	1.9883 1.9883 2.9090	0.8835 0.8835 1.0064
	40^2	1.6464e−4 1.6464e−4 1.8808e−4	1.9941 1.9941 1.9963	7.8949e−5 7.8949e−5 2.2516e−6	1.9899 1.9899 2.8739	0.8818 0.8818 1.0049
2	20^2	1.8615e−6 1.8615e−6 2.1636e−6	–	9.2510e−7 9.2510e−7 4.3112e−8	–	0.8738 0.8738 1.0068
	30^2	5.5275e−7 5.5275e−7 6.4190e−7	2.9946 2.9946 2.9968	2.7502e−7 2.7502e−7 8.5587e−9	2.9918 2.9918 3.9877	0.8714 0.8714 1.0045
	40^2	2.3346e−7 2.3346e−7 2.7097e−7	2.9960 2.9960 2.9978	1.1626e−7 1.1626e−7 2.7180e−9	2.9929 2.9929 3.9872	0.8702 0.8702 1.0034
3	20^2	4.0497e−9 4.0497e−9 4.7423e−9	–	2.0511e−9 2.0511e−9 1.2019e−10	–	0.8674 0.8674 1.0056
	30^2	8.0124e−10 8.0124e−10 9.3760e−10	3.9960 3.9960 3.9978	4.0607e−10 4.0607e−10 1.5854e−11	3.9945 3.9945 4.9960	0.8655 0.8655 1.0038
	40^2	2.5374e−10 2.5374e−10 2.9680e−10	3.9969 3.9969 3.9984	1.2867e−10 1.2867e−10 3.7661e−12	3.9949 3.9948 4.9964	0.8644 0.8644 1.0029

Table 2
$L^2(\Omega)$-errors $\|\mathbf{e}\|_{2,\Omega}$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|_{2,\Omega}$ and their order of convergence. Global effectivity indices corresponding to transient estimates for Example 5.1 at $t=1$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|$	Order	θ
1	10^2	4.7312e−3	–	2.1562e−4	–	1.014
	20^2	1.1920e−3	1.989	3.7742e−5	2.514	1.007
	30^2	5.3133e−4	1.993	1.4545e−5	2.352	1.005
	40^2	2.9931e−4	1.995	7.5490e−6	2.28	1.004
2	10^2	2.7124e−5	–	1.3917e−6	–	1.007
	20^2	3.4076e−6	2.993	1.2010e−7	3.535	1.003
	30^2	1.0115e−6	2.996	3.0819e−8	3.355	1.002
	40^2	4.2712e−7	2.997	1.1987e−8	3.283	1.001
3	10^2	1.1855e−7	–	7.7486e−9	–	1.005
	20^2	7.4357e−9	3.995	3.0928e−10	4.647	1.002
	30^2	1.4707e−9	3.997	5.0845e−11	4.453	1.001
	40^2	4.6568e−10	3.998	1.4522e−11	4.356	1.000

We choose the initial and boundary conditions, such that the true solution is

$$\mathbf{u}=(\,-4\quad 5\quad -1\quad 2\quad 1\quad -3\,)^t\cos\left(t+\frac{x_1+x_2+x_3}{\sqrt{3}}\right), \tag{5.6d}$$

The matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ each have eigenvalues $\{-1,-1,0,0,1,1\}$. Since $\bigcap_{i=1}^{3}\mathcal{N}(\mathbf{A}_i)^{\perp}=\{\mathbf{0}\}$, the stationary error estimate $\mathbf{E}^{\mathcal{R}}$ will not be an accurate estimate of any component of the error.

We solve (5.6) on uniform meshes having $N=10^3, 15^3, 20^3$ elements with $p=1,2,3$. The results for transient error estimates shown in Table 3 suggest that the global effectivity indices for the transient error estimate converge to unity under mesh refinement. Furthermore, we plot the effectivity indices for the transient error estimate versus time in Fig. 3 to show that $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ is asymptotically exact at all times, which is in full agreement with Theorem 4.3. Thus, the transient effectivity indices stay close to unity.

Example 5.3. Let us consider the acoustic wave equation,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho_0\mathbf{v})=0,\qquad \frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\frac{c_0^2}{\rho_0}\rho\right)=\mathbf{0}, \tag{5.7}$$

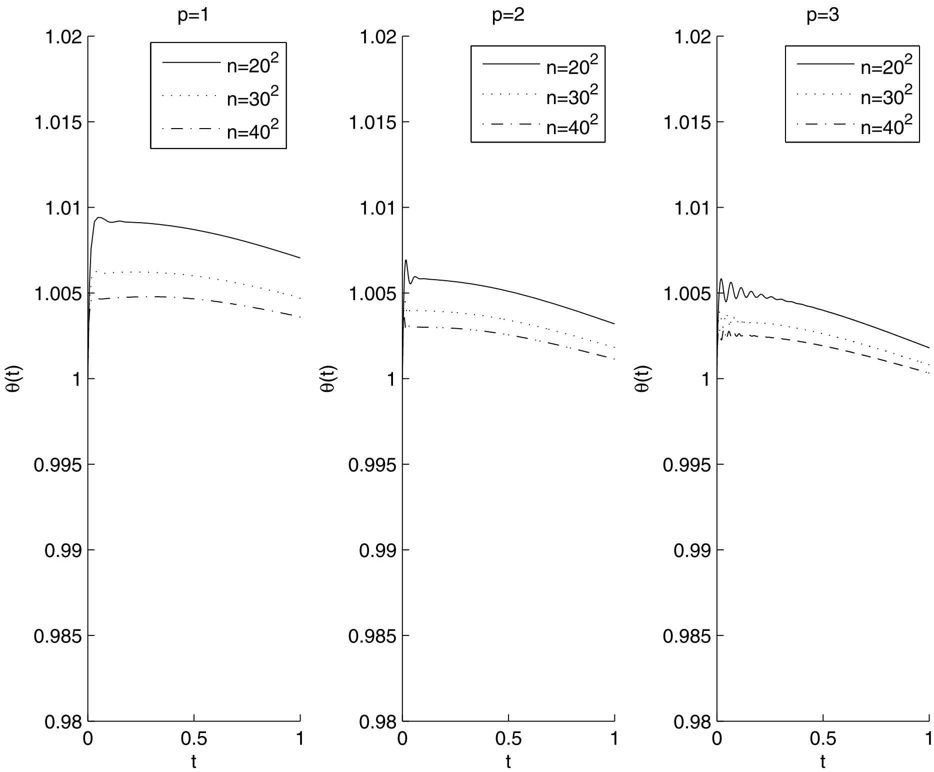

Fig. 2. Global transient effectivity indices versus time for Example 5.1.

Table 3
L^2-errors $\|\mathbf{e}\|_{2,\Omega}$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|_{2,\Omega}$, their order of convergence and global transient effectivity indices θ for Example 5.2 at $t=1$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|$	Order	θ
1	10^3	7.0837e−4	–	1.1846e−4	–	0.9679
	15^3	3.2053e−4	1.956	4.4943e−5	2.39	0.9678
	20^3	1.8214e−4	1.965	2.3108e−5	2.312	0.9698
2	10^3	7.1585e−6	–	3.4687e−7	–	0.9918
	15^3	2.1201e−6	3.001	8.5545e−8	3.453	0.9937
	20^3	8.9429e−7	3	3.2401e−8	3.375	0.9948
3	10^3	1.2723e−8	–	4.9097e−9	–	0.8902
	15^3	2.4589e−9	4.054	6.9491e−10	4.822	0.9245
	20^3	7.7459e−10	4.015	1.7758e−10	4.743	0.9386

where ρ is the density, $\mathbf{v}=(v_1,v_2)^t$ is the velocity field, ρ_0 is the reference density, c_0 is the speed of sound. In two space dimensions,if we choose space and time units such that $c_0=1$, $\rho_0=1$, (5.7) can be written as the symmetric hyperbolic system

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{A}_1\frac{\partial \mathbf{u}}{\partial x_1}+\mathbf{A}_2\frac{\partial \mathbf{u}}{\partial x_2}=\mathbf{0}, \quad \mathbf{x}\in\Omega=(0,1)^2,\ 0<t<1, \tag{5.8a}$$

where

$$\mathbf{u}=\begin{pmatrix}\rho\\ v_1\\ v_2\end{pmatrix}, \qquad \mathbf{A}_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \qquad \mathbf{A}_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}. \tag{5.8b}$$

We select initial and boundary conditions such that the true solution is

$$\rho(t,\mathbf{x})=\sin\left(t-\frac{x+y}{\sqrt{2}}\right), \qquad \mathbf{v}(t,\mathbf{x})=\frac{1}{\sqrt{2}}\sin\left(t-\frac{x+y}{\sqrt{2}}\right)(1,1)^t. \tag{5.8c}$$

Both matrices $\mathbf{A}_1$, $\mathbf{A}_2$ are singular and admit the eigenvalues $\{-1,0,1\}$. Moreover, the eigenvectors $(0,1,0)^t$ and $(0,0,1)^t$ are associated with the zero eigenvalue for $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Applying our theory, the stationary error estimate $\mathbf{E}^{\mathcal{R}}$

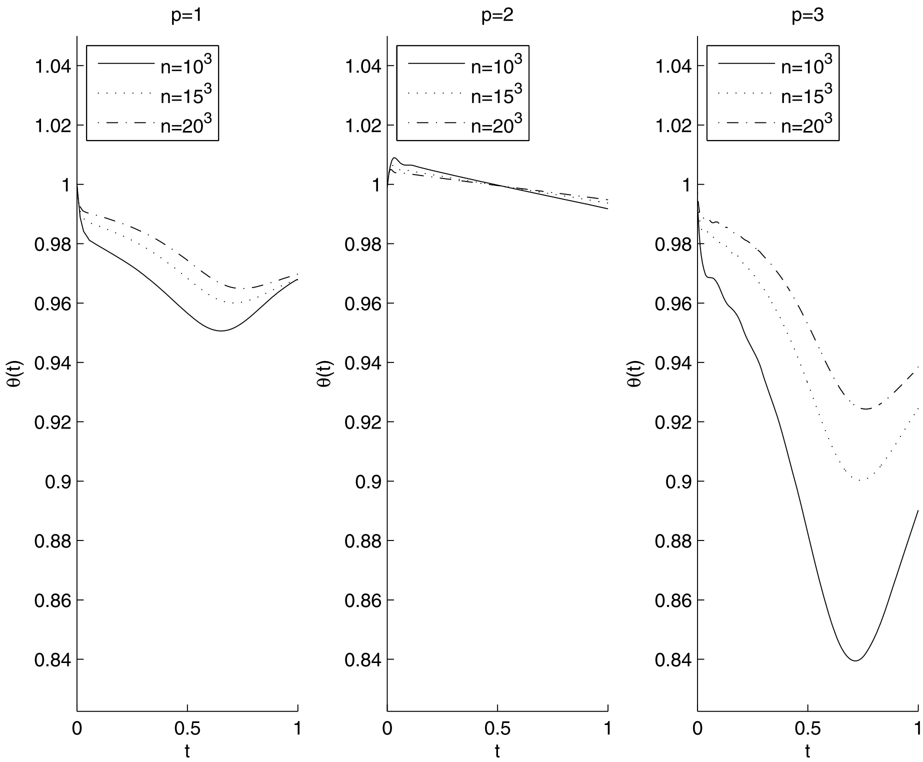

Fig. 3. Global transient effectivity indices versus time for Example 5.2.

Table 4
L^2-errors $\|\mathbf{e}\|_{2,\Omega}$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|_{2,\Omega}$ and their order of convergence. Global effectivity indices corresponding to transient estimates for Example 5.3 at $t=1$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|$	Order	θ
1	10^2	2.4782e−4	–	3.5072e−5	–	0.9685
	20^2	6.3394e−5	1.967	6.5200e−6	2.427	0.9717
	30^2	2.8435e−5	1.977	2.5723e−6	2.294	0.974
2	10^2	2.9606e−6	–	1.1041e−7	–	0.9927
	20^2	3.6975e−7	3.001	1.0393e−8	3.409	0.9956
	30^2	1.0953e−7	3	2.7126e−9	3.313	0.9967
3	10^2	6.2331e−9	–	1.0456e−9	–	0.9562
	20^2	3.9372e−10	3.985	4.5621e−11	4.518	0.9695
	30^2	7.8144e−11	3.988	7.8266e−12	4.348	0.9748

can only accurately approximate the component of the error lying in $(\mathcal{N}(\mathbf{A}_1)\oplus\mathcal{N}(\mathbf{A}_2))^{\perp}=\text{span}\{(1,0,0)^t\}$, *i.e.*, only $E_1^{\mathcal{R}}$ is an accurate estimate of e_1.

We solve (5.8) on uniform meshes having $N=10^2,20^2,30^2$ elements for $p=1,2,3$ and present the L^2-errors and effectivity indices for the transient error estimate $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ at $t=1$ in Table 4. We observe that the effectivity indices converge to unity under mesh refinement. In Fig. 4 we plot the global effectivity indices for the transient error estimate versus time. We observe that the error estimate $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ is asymptotically exact at all $0\leqslant t\leqslant 1$ under mesh refinement which is in full agreement with Theorem 4.3. Thus, the transient effectivity indices stay close to unity at $0\leqslant t\leqslant 1$.

The longtime behavior of our error estimates is shown by solving (5.8) for $0\leqslant t\leqslant 20$ and $p=1$. We present the errors and effectivity indices at $t=20$ in Table 5 and plot the $L2$-errors and effectivity indices versus time in Fig. 5. Our procedure yields reasonable error estimates with effectivity indices staying near unity for $0\leqslant t\leqslant 20$.

The previous computations suggest that the global error estimate $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ is not $O(h^{p+2})$. however, it provides an accurate estimate of the DG error in all components of the solution. At this point we do not have a rigorous explanation for this phenomenon.

Example 5.4. Many problems in acoustics and electromagnetism involve reflection boundary conditions and the interaction of incident and reflected waves. In order to show the efficiency of the proposed error estimates for these problems we

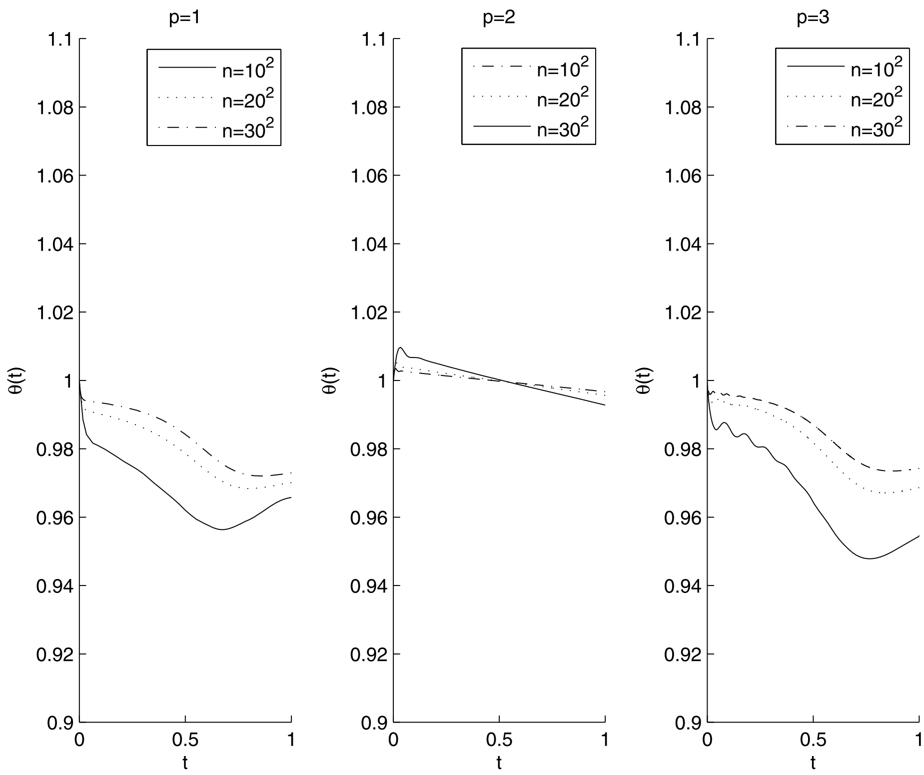

Fig. 4. Global transient effectivity indices versus time for Example 5.3.

Table 5
L^2-errors and orders of convergence with transient global effectivity indices for Example 5.3 at $t = 20$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e} - \mathbf{E}^{\mathcal{R}} - \mathbf{E}^{\mathcal{N}}\|$	Order	θ
1	10^2	2.8052e−4	–	4.3635e−5	–	1.043
	20^2	7.1519e−5	1.972	7.1738e−6	2.605	1.000
	30^2	3.2275e−5	1.962	3.7026e−6	1.631	0.9763

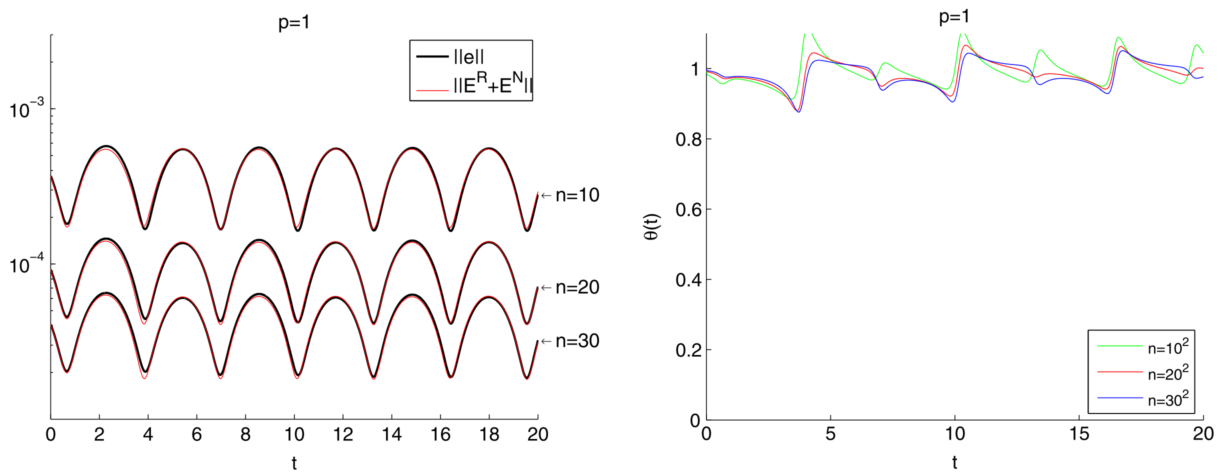

Fig. 5. L^2-errors (left) and global transient effectivity indices (right) versus time for Example 5.3 with $p = 1$.

consider the 2D acoustics problem (5.8) on the square domain $[-2, 0] \times [-1, 1]$ and $0 < t \leqslant 1$. We apply Dirichlet boundary conditions (3.1c) on the left, top and bottom boundaries while at the right boundary $x = 0$ we apply the reflection boundary condition by providing an external state as

Table 6
Componentwise L^2-errors $\|\mathbf{e}\|_{2,\Omega}$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|_{2,\Omega}$ and their order of convergence. Global effectivity indices corresponding to transient estimates for Example 5.4 at $t=1$.

p	N	$\|\mathbf{e}\|^*$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|^*$	Order	θ^*
1	20^2	$\begin{bmatrix}7.4742e-03\\6.1771e-03\\6.1692e-03\end{bmatrix}$	–	$\begin{bmatrix}4.3337e-03\\3.1275e-03\\3.0394e-03\end{bmatrix}$	–	$\begin{bmatrix}0.8624\\0.8950\\0.8999\end{bmatrix}$
	40^2	$\begin{bmatrix}1.7046e-03\\1.4342e-03\\1.4419e-03\end{bmatrix}$	$\begin{bmatrix}2.1325\\2.1067\\2.0971\end{bmatrix}$	$\begin{bmatrix}5.6951e-04\\4.0721e-04\\3.9346e-04\end{bmatrix}$	$\begin{bmatrix}2.9278\\2.9412\\2.9495\end{bmatrix}$	$\begin{bmatrix}0.9527\\0.9667\\0.9657\end{bmatrix}$
	60^2	$\begin{bmatrix}7.3991e-04\\6.2573e-04\\6.3033e-04\end{bmatrix}$	$\begin{bmatrix}2.0583\\2.0456\\2.0408\end{bmatrix}$	$\begin{bmatrix}1.6997e-04\\1.2186e-04\\1.1691e-04\end{bmatrix}$	$\begin{bmatrix}2.9821\\2.9755\\2.9931\end{bmatrix}$	$\begin{bmatrix}0.9772\\0.9851\\0.9823\end{bmatrix}$
	80^2	$\begin{bmatrix}4.1247e-04\\3.4951e-04\\3.5238e-04\end{bmatrix}$	$\begin{bmatrix}2.0313\\2.0244\\2.0214\end{bmatrix}$	$\begin{bmatrix}7.1838e-05\\5.1812e-05\\4.9347e-05\end{bmatrix}$	$\begin{bmatrix}2.9936\\2.9729\\2.9981\end{bmatrix}$	$\begin{bmatrix}0.9867\\0.9922\\0.9886\end{bmatrix}$
2	20^2	$\begin{bmatrix}2.7232e-04\\2.8163e-04\\2.8100e-04\end{bmatrix}$	–	$\begin{bmatrix}8.9865e-05\\1.7760e-04\\1.7888e-04\end{bmatrix}$	–	$\begin{bmatrix}0.9686\\0.7749\\0.7630\end{bmatrix}$
	40^2	$\begin{bmatrix}3.3295e-05\\3.6141e-05\\3.6024e-05\end{bmatrix}$	$\begin{bmatrix}3.0319\\2.9621\\2.9636\end{bmatrix}$	$\begin{bmatrix}5.7188e-06\\2.3159e-05\\2.3416e-05\end{bmatrix}$	$\begin{bmatrix}3.9740\\2.9390\\2.9334\end{bmatrix}$	$\begin{bmatrix}0.9925\\0.7529\\0.7439\end{bmatrix}$
	60^2	$\begin{bmatrix}9.8232e-06\\1.0797e-05\\1.0758e-05\end{bmatrix}$	$\begin{bmatrix}3.0105\\2.9796\\2.9807\end{bmatrix}$	$\begin{bmatrix}1.1371e-06\\6.8981e-06\\6.9692e-06\end{bmatrix}$	$\begin{bmatrix}3.9838\\2.9870\\2.9890\end{bmatrix}$	$\begin{bmatrix}0.9963\\0.7509\\0.7428\end{bmatrix}$
	80^2	$\begin{bmatrix}4.1379e-06\\4.5730e-06\\4.5558e-06\end{bmatrix}$	$\begin{bmatrix}3.0053\\2.9863\\2.9867\end{bmatrix}$	$\begin{bmatrix}3.6080e-07\\2.9102e-06\\2.9373e-06\end{bmatrix}$	$\begin{bmatrix}3.9901\\2.9999\\3.0034\end{bmatrix}$	$\begin{bmatrix}0.9977\\0.7526\\0.7448\end{bmatrix}$
3	20^2	$\begin{bmatrix}1.0468e-05\\9.5103e-06\\9.4296e-06\end{bmatrix}$	–	$\begin{bmatrix}5.2319e-06\\5.6794e-06\\5.6357e-06\end{bmatrix}$	–	$\begin{bmatrix}0.8925\\0.8113\\0.8129\end{bmatrix}$
	40^2	$\begin{bmatrix}6.1655e-07\\5.4138e-07\\5.3651e-07\end{bmatrix}$	$\begin{bmatrix}4.0856\\4.1348\\4.1355\end{bmatrix}$	$\begin{bmatrix}1.6972e-07\\2.1422e-07\\2.1000e-07\end{bmatrix}$	$\begin{bmatrix}4.9461\\4.7286\\4.7461\end{bmatrix}$	$\begin{bmatrix}0.9670\\0.9166\\0.9187\end{bmatrix}$
	60^2	$\begin{bmatrix}1.2021e-07\\1.0511e-07\\1.0427e-07\end{bmatrix}$	$\begin{bmatrix}4.0321\\4.0427\\4.0400\end{bmatrix}$	$\begin{bmatrix}2.2577e-08\\3.5508e-08\\3.4757e-08\end{bmatrix}$	$\begin{bmatrix}4.9751\\4.4326\\4.4362\end{bmatrix}$	$\begin{bmatrix}0.9845\\0.9395\\0.9410\end{bmatrix}$
	80^2	$\begin{bmatrix}3.7854e-08\\3.3070e-08\\3.2832e-08\end{bmatrix}$	$\begin{bmatrix}4.0167\\4.0195\\4.0169\end{bmatrix}$	$\begin{bmatrix}5.3805e-09\\1.0454e-08\\1.0253e-08\end{bmatrix}$	$\begin{bmatrix}4.9852\\4.2502\\4.2437\end{bmatrix}$	$\begin{bmatrix}0.9910\\0.9476\\0.9486\end{bmatrix}$
4	20^2	$\begin{bmatrix}4.1137e-07\\4.1926e-07\\4.1992e-07\end{bmatrix}$	–	$\begin{bmatrix}2.7349e-07\\3.3860e-07\\3.3763e-07\end{bmatrix}$	–	$\begin{bmatrix}0.7766\\0.6355\\0.6364\end{bmatrix}$
	40^2	$\begin{bmatrix}1.0911e-08\\1.1334e-08\\1.1388e-08\end{bmatrix}$	$\begin{bmatrix}5.2365\\5.2091\\5.2045\end{bmatrix}$	$\begin{bmatrix}4.3713e-09\\7.8820e-09\\7.9295e-09\end{bmatrix}$	$\begin{bmatrix}5.9673\\5.4249\\5.4121\end{bmatrix}$	$\begin{bmatrix}0.9283\\0.7230\\0.7217\end{bmatrix}$
	60^2	$\begin{bmatrix}1.3819e-09\\1.4222e-09\\1.4307e-09\end{bmatrix}$	$\begin{bmatrix}5.0963\\5.1191\\5.1161\end{bmatrix}$	$\begin{bmatrix}3.8644e-10\\9.0686e-10\\9.1549e-10\end{bmatrix}$	$\begin{bmatrix}5.9829\\5.3330\\5.3245\end{bmatrix}$	$\begin{bmatrix}0.9660\\0.7603\\0.7588\end{bmatrix}$
	80^2	$\begin{bmatrix}3.2316e-10\\3.3110e-10\\3.3331e-10\end{bmatrix}$	$\begin{bmatrix}5.0509\\5.0664\\5.0642\end{bmatrix}$	$\begin{bmatrix}6.9042e-11\\2.0127e-10\\2.0339e-10\end{bmatrix}$	$\begin{bmatrix}5.9867\\5.2327\\5.2292\end{bmatrix}$	$\begin{bmatrix}0.9803\\0.7783\\0.7767\end{bmatrix}$

$$(\nu_i \mathbf{A}_i)^- \mathbf{u}_h^- = (\nu_i \mathbf{A}_i)^- \mathbf{u}_h^{ext}, \tag{5.9a}$$

where

$$\mathbf{u}_h^{ext} = \begin{pmatrix} \rho_h^+ \\ -v_{1,h}^+ \\ v_{2,h}^+ \end{pmatrix}. \tag{5.9b}$$

The initial and Dirichlet boundary conditions are selected such that the true solution is the sum of an incident and a reflected waves

$$\mathbf{u} = \mathbf{u}^i + \mathbf{u}^r, \tag{5.10a}$$

where the incident wave is a plane Gaussian propagating from left to right at a 45° angle and is given by

$$\mathbf{u}^i(t,x,y) = \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \psi\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - (t-0.5)\right). \tag{5.10b}$$

After hitting the left boundary $x=0$ it reflects as a plane Gaussian wave given by

Table 7
L^2-errors $\|\mathbf{e}\|_{2,\Omega}$, $\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|_{2,\Omega}$ and their order of convergence. Global effectivity indices corresponding to transient estimates for Example 5.4 at $t=1$.

p	N	$\|\mathbf{e}\|$	Order	$\|\mathbf{e}-\mathbf{E}^{\mathcal{R}}-\mathbf{E}^{\mathcal{N}}\|$	Order	θ
1	20^2	1.1493e−02	–	6.1482e−03	–	0.8828
	40^2	2.6536e−03	2.115	8.0310e−04	2.936	0.9606
	60^2	1.1560e−03	2.049	2.3960e−04	2.983	0.981
	80^2	6.4533e−04	2.026	1.0139e−04	2.989	0.9889
2	20^2	4.8212e−04	–	2.6761e−04	–	0.8379
	40^2	6.0930e−05	2.984	3.3427e−05	3.001	0.8289
	60^2	1.8133e−05	2.989	9.8715e−06	3.008	0.8279
	80^2	7.6674e−06	2.992	4.1506e−06	3.012	0.829
3	20^2	1.6998e−05	–	9.5598e−06	–	0.8435
	40^2	9.8035e−07	4.116	3.4467e−07	4.794	0.9375
	60^2	1.9071e−07	4.038	5.4576e−08	4.545	0.9581
	80^2	6.0037e−08	4.018	1.5600e−08	4.353	0.9654
4	20^2	7.2204e−07	–	5.5086e−07	–	0.6848
	40^2	1.9422e−08	5.216	1.2005e−08	5.52	0.7932
	60^2	2.4452e−09	5.111	1.3453e−09	5.398	0.8311
	80^2	5.7022e−10	5.061	2.9435e−10	5.282	0.8479

$$\mathbf{u}^r(t,x,y)=\begin{pmatrix}1\\ -1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\psi\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-(t-0.5)\right), \tag{5.10c}$$

with $\psi(z)=e^{-10z^2}$.

We solve this problem using uniform meshes having $N=20^2,40^2,60^2,80^2$ square elements and $p=1,2,3,4$. We integrate from $t=0$ to $t=1$ with a temporal tolerance such that the spatial errors dominate the total discretization errors and present the L^2-errors and effectivity indices for the transient error estimate $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ at $t=1$ in Tables 6 and 7. We observe that the effectivity indices converge to unity under mesh refinement for $p=1,3$ which suggests that our error estimates $\mathbf{E}^{\mathcal{R}}+\mathbf{E}^{\mathcal{N}}$ are asymptotically correct under mesh refinement. We further observe that our estimates are more accurate for odd degrees $p=1$ and $p=3$ while for even degrees $p=2$ and $p=4$ the error estimates for the density are more accurate than those for the velocity at least for the meshes considered here. The global effectivity indices versus time shown in Fig. 6 suggest that the proposed error estimates stay accurate for all times, $0<t<1$. From this example we recommend our error estimates for problems with reflection boundary conditions and odd-degree polynomial spaces.

6. Conclusions

In this paper, we investigated the DG method for first-order linear symmetric hyperbolic systems with the enriched polynomial space $\mathcal{P}_p$, $\mathbb{P}_p\subset\mathcal{P}_p\subset\mathbb{P}_{p+1}$, and modified L^2-projections to approximate the initial and boundary conditions. We performed a local error analysis where we showed that the leading term of the discretization error lies in a polynomial subspace spanned by a linear combination of Legendre polynomials of order p and $p+1$. For systems with nonsingular coefficient matrices the leading term of the error is spanned by $(p+1)$th-degree Radau polynomials. We also established that a projection of the DG error is $\mathcal{O}(h^{p+2})$ superconvergent at Radau points.

We applied the results on the asymptotic behavior of the DG error to construct efficient and asymptotically exact implicit residual-based *a posteriori* error estimates. We split the error into two parts and estimated each part separately by solving a small system of equations based on the local residual of the PDE. Thus, we were able to compute an efficient estimate of the discretization error by solving a local problem on each element. For systems with invertible coefficient matrices, the error estimates are obtained by solving a stationary problem, while, for general systems, part of the error is computed by solving a local transient system of equations. A local error analysis was performed to show that, for smooth solutions, the error estimates are asymptotically correct. Finally, we confirmed our results numerically for the acoustic problem and Maxwell's equations.

We note that we are not able to prove the asymptotic exactness of our global *a posteriori* error estimates for hyperbolic systems. However, the computational results in this paper suggest that global *a posteriori* error estimates are asymptotically exact for smooth solutions. Guided by our analysis for the one-dimensional scalar kinematic wave equation [4], we plan to prove the convergence of the global error estimate in the future. Several challenges such as nonlinear hyperbolic systems, other numerical fluxes and unstructured meshes remain. We plan to investigate the behavior of DG errors with Lax–Friedrichs and Roe fluxes [11]. In order to solve general problems with curved geometry in acoustics and electromagnetism we plan to investigate the extension of the work of Adjerid and Baccouch [2,3] to linear and nonlinear hyperbolic systems on unstructured meshes.

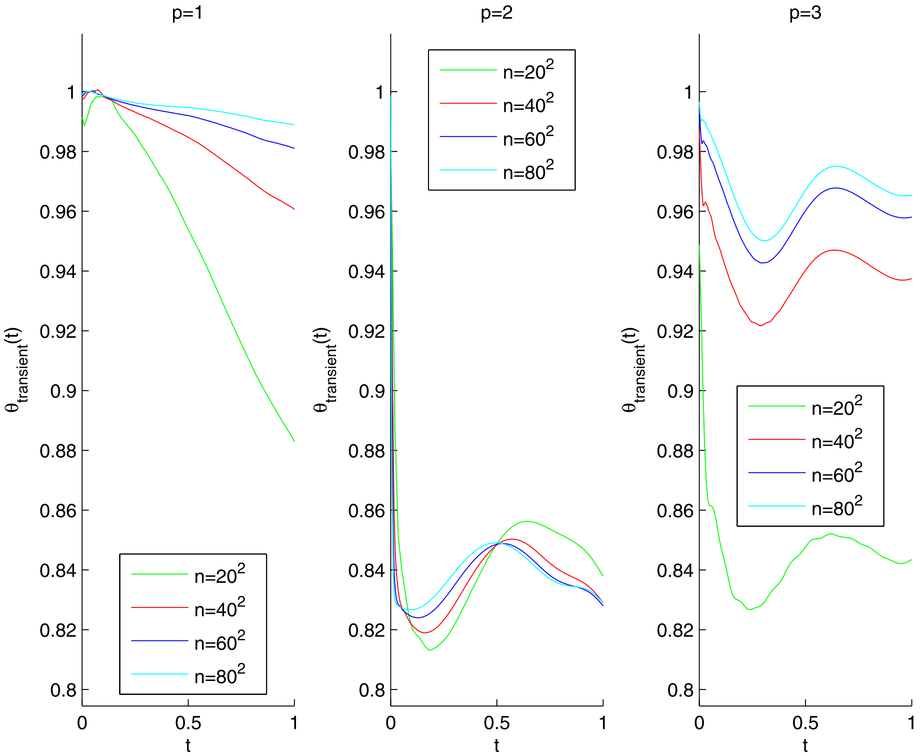

Fig. 6. Global transient effectivity indices versus time for Example 5.4.

Acknowledgement

This research was partially supported by the National Science Foundation (Grant Numbers DMS 0511806, DMS 0809262).

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